Alternative and Robust Technique for the Calculation of Dispersion Relations in Grounded Layered Media

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1. Introduction

The study of the modal characteristics of grounded layered media is extremely important in the analysis of commonly employed microwave and optics circuits [1]. In a general case, this task requires the root searching of certain functions in the 2D complex plane, which provides the modal solutions of the structure under analysis. Different techniques have been proposed in the literature to solve this problem, ranging from using initial quasi-static asymptotic points for the complex search [2] to methods of integral nature [3], [4]. However, still a simple and efficient search strategy is not generally found.

In [5], a novel approach was proposed for the computation of complex modes related to laterally shielded multilayered media. In this contibution, we employ the underlying ideas introduced in [5] to obtain the dispersion relations of grounded unbounded layered media. The proposed technique is based on the relations between modes of closed and open layered media, which allow to track the modes in the complex plane when the boundary conditions of the structure are modified from PEC to free-space. The main advantage of this approach is that, as in the case of integral methods [3] and unlikely purely differential techniques [6], it allows to compute all complex modes within a given region under study. Furthermore, the method is easy to implement, fast, and provides insight into the nature of complex modes.

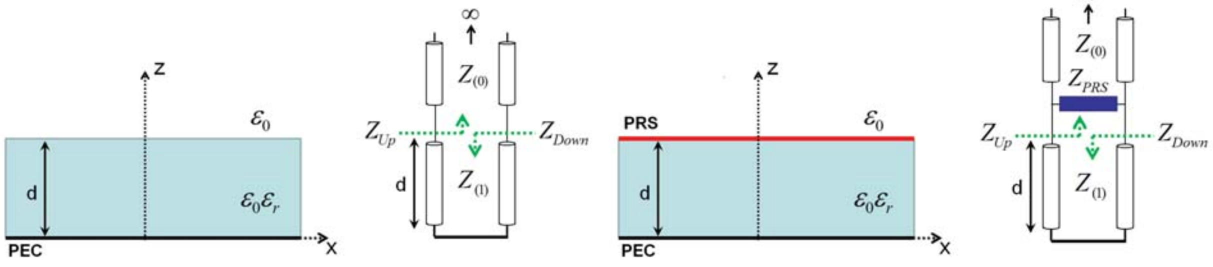
2. Proposed Root Searching Method

The transverse resonance equation (TRE) [1] is a standard method able to compute the dispersion relations of many different structures, including isotropic and anisotropic slabs [7], [8] or complex modes appearing in leaky-wave antennas [9]. In the particular case of grounded slabs (see Fig. 1a-b), the dispersion relation can be obtained by solving the equation

$$Z_{Up}^{TE,TM} + Z_{Down}^{TE,TM} = 0, \tag{1}$$

where $Z_{Down}^{TE,TM} = jZ_{(0)}^{TE,TM}\tan(k_{z_0}d)$ and $Z_{Up}^{TE,TM} = Z_{(0)}^{TE,TM}$ are the input impedance of the slab (see Fig. 1b) and the characteristic impedance of free space, respectively. Also, by $Z^{TM,TE}$ we denote the TM$_z$ or TE$_z$ characteristic impedance of a medium, where $Z^{TM} = k_z/(\omega\varepsilon_0\varepsilon_r)$ and $Z^{TE} = \omega\mu_0/k_z$, being $k_z = \pm\sqrt{\varepsilon_r k_0^2 - k_\rho^2}$ the longitudinal complex propagation constant (normal to the slab).

The roots of Eq. 1 provides the TM$_z$ and TE$_z$ complex modes of the structure. As previously commented, the solution of this equation requires a search of the modes in the complex 2D plane, which is usually performed by applying the Newton-Raphson algorithm [6]. However, this algorithm highly depends on the starting point during the search and it does not guarantee that all modes of a given region are found. In order to overcome these problems, we propose a novel approach based on relating the modes of closed and open layered media [5]. The technique employs an alternative structure (see Fig. 1c), composed by a partially reflective surface (PRS) covering the grounded isotropic slab, and it tracks the evolution of the modes in the complex plane when the structure is modified from a parallel-plate waveguide (Z_{PRS} set as a short-circuit) to an to an open grounded slab (Z_{PRS} set as an open-circuit).



(a) (b) (c) (d)

Figure 1: Schematic of an open grounded isotropic slab (a) and its corresponding transmission line model along the *z*-axis (b). Alternative structure (c), composed of a partially reflective surface (PRS) covering a grounded isotropic slab, proposed to study the dispersion relation of (a). The modification of the PRS surface impedance, Z_{PRS}, allows to model different structures, ranging from a closed parallel-plate waveguide (PPW) to an open grounded slab. (d) Transmission line equivalent circuit of the structure.

The details of the proposed algorithm are as follows. First, the PRS impedance is set to a short-circuit (Z_{PRS}=0), transforming the structure of Fig. 1c into a parallel-plate waveguide (PPW). The modes of this guide, which are propagative (purely positive) and evanescent (purely imaginary), are known analytically [1]

$$k_\rho^{TE,TM} = \sqrt{\varepsilon_r k_0^2 - \left(\frac{\mathrm{m}\pi}{d}\right)}, \qquad \mathrm{m} = 0, 1, 2\ldots \tag{2}$$

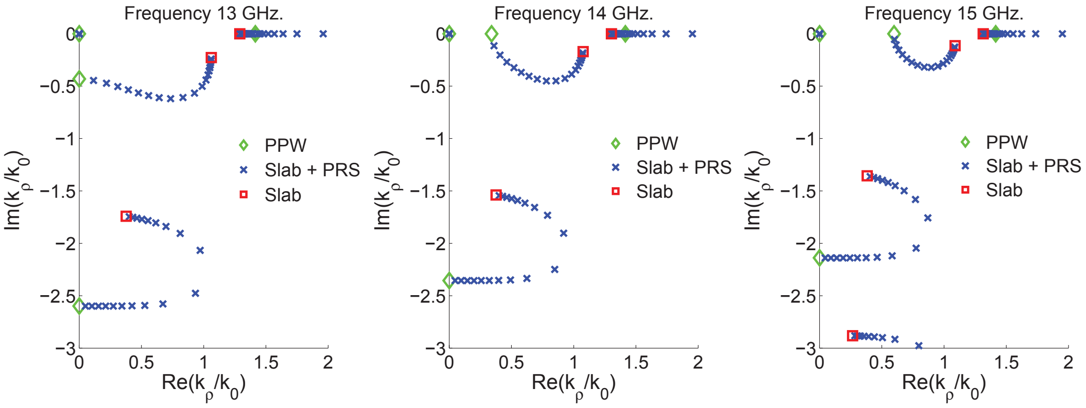
Then, the impedance of the PRS is increased. This situation is modeled by the equivalent transmission line circuit shown in Fig. 1d. In this case, the impedances of the figure are defined as $Z_{Down}^{TE,TM} = jZ_{(1)}^{TE,TM}\tan(k_z d)$ and $Z_{Up}^{TE,TM} = \frac{Z_{(0)}^{TE,TM} Z_{PRS}}{Z_{(0)}^{TE,TM}+Z_{PRS}} = \xi(Z_{PRS})Z_{(0)}^{TE,TM}$, where the term $\xi(Z_{PRS})$ has been included for convenience and varies from 0 to 1 as Z_{PRS} goes from a short to an open circuit. The imposition of the TRE (see Eq. 1) on the transmission line model of Fig. 1d allows to obtain the following equations for the TM$_z$ and TE$_z$ modes

$$j\frac{\sqrt{\varepsilon_r k_0^2 - k_\rho^{TM^2}}}{\omega\varepsilon_0\varepsilon_r}\tan\left(\sqrt{\varepsilon_r k_0^2 - k_\rho^{TM^2}}\,d\right) \pm \xi(Z_{PRS})\frac{\sqrt{k_0^2 - k_\rho^{TM^2}}}{\omega\varepsilon_0} = 0, \tag{3}$$

$$j\frac{\omega\mu_0}{\sqrt{\varepsilon_r k_0^2 - k_\rho^{TE^2}}}\tan\left(\sqrt{\varepsilon_r k_0^2 - k_\rho^{TE^2}}\,d\right) \pm \xi(Z_{PRS})\frac{\omega\mu_0}{\sqrt{k_0^2 - k_\rho^{TE^2}}} = 0. \tag{4}$$

Note that these equations reduce to the standard TREs of a grounded slab for the case of $\xi(Z_{PRS}) = 1$. Eq.3 and Eq.4 are then solved by applying the Newton-Raphson algorithm [6], taking as starting points for the search the modes computed analytically for the closed structure. This procedure repeated by increasing at each iteration the impedance of the PRS and taking as starting points for the search in the 2D complex plane the solutions obtained at the previous step. In the final step of the algorithm, when Z_{PRS} is an open circuit, the computed modes will correspond to the desired grounded slab structure.

It is interesting to note that initial evanescent and propagating modes of the closed PPW ($Z_{PRS} = 0$) move into the complex plane if the PRS impedance of the structure is greater than zero, *i.e.*, when some energy may start to leak to free space. Also, note that the tracking of the modes in the complex plane uniquely determines their number and position in a given region. Moreover, note that this technique is computationally efficient and requires few iterations to converge. It also provides the important advantage of knowing if the tracking procedure is accurate. Specifically, if one or more modes are missing in a particular iteration, it is simple to modify the value of the PRS impedance to re-start the tracking from a previous situation. This feature makes the proposed algorithm robust and stable.



(a) (b) (c)

Figure 2: Evolution of normalized TM$_z$ modes in the complex plane, related to the structure shown in Fig. 1c and computed at several frequencies, versus different PRS impedances. The range of impedances varies from a short circuit (which represents a PPW medium, shown by a diamond marker) to an open circuit (which represents an open grounded slab, indicated by a square marker), including intermediate values (depicted by an 'x' marker) employed to track the modes evolution.

3. Dispersion Analysis of an Open Grounded Slab and Discussion

In this Section we apply the theory developed in Section II for the computation of the dispersion relations of a grounded slab (see Fig. 1a), with parameters $\varepsilon_r = 2$, $\mu_r = 1$ and $d = 7.8$ mm.

Fig 2 and Fig 3 present detailed examples of the proposed algorithm for the computation of TM$_z$ and TE$_z$ modes. First, the modes are analytically computed for the case of a PPW (indicated by diamond markers), which corresponds to setting Z_{PRS} as a short-circuit. Then, the figures show the evolution of the modes (indicated by 'x' markers) in the complex k_ρ plane when the structure is progressively transformed from a PPW to an open grounded slab. Note that, for the sake of the method's clarity and to show the modes evolution, we have employed an unnecessary large number of intermediate steps to compute these results. The final solution of the problem is shown by using square markers. As can be observed in the figures, each individual mode is easily tracked in the complex plane. In both TM$_z$ and TE$_z$ cases, the evanescent modes of the PPW are transformed into improper leaky-waves of the slab (where $\mathrm{Re}[k_\rho/k_0]<1$ and $\mathrm{Im}[k_\rho/k_0]<0$ [7]). When the frequency increases, an evanescent mode of the PPW becomes eventually propagative, now being related to a leaky-mode of the slab. If we increase more the frequency, the propagative PPW modes are transformed into surface modes or to a non-physical improper modes of the slab, which present both a purely real propagation constant.

Finally, Fig. 4 presents the TM$_z$ and TE$_z$ dispersion curves $[\mathrm{Re}(k_\rho/k_0)]$ of the grounded slab under study. The results shown in the figure are in excellent agreement with those found in literature [7], fully demonstrating that the the proposed technique is able to compute all modes, including real proper (surface modes), non-physical improper and complex improper (leaky modes), in a systematic, robust, and efficient fashion.

Acknowledgments

This work was supported by the Swiss National Science Foundation under grant number 133583, and by the FP7 IEF project "Marconi", ref. PIEF-GA-2011 – 300966.

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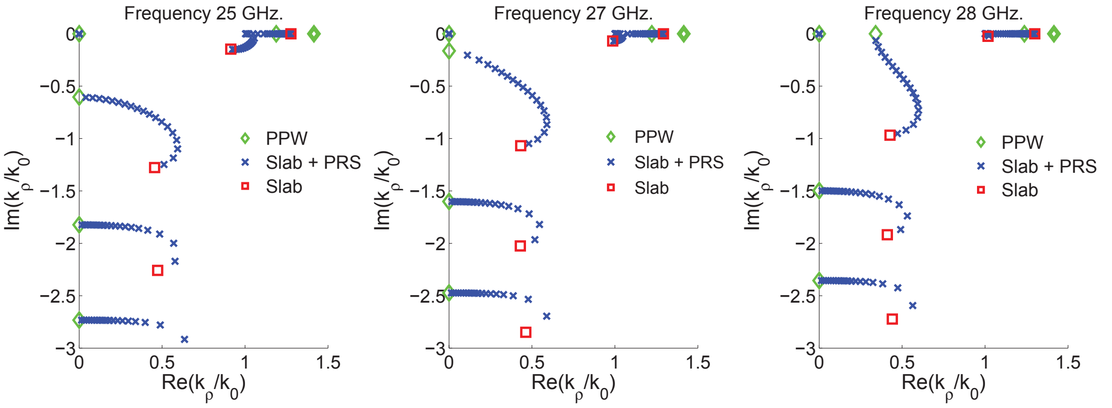
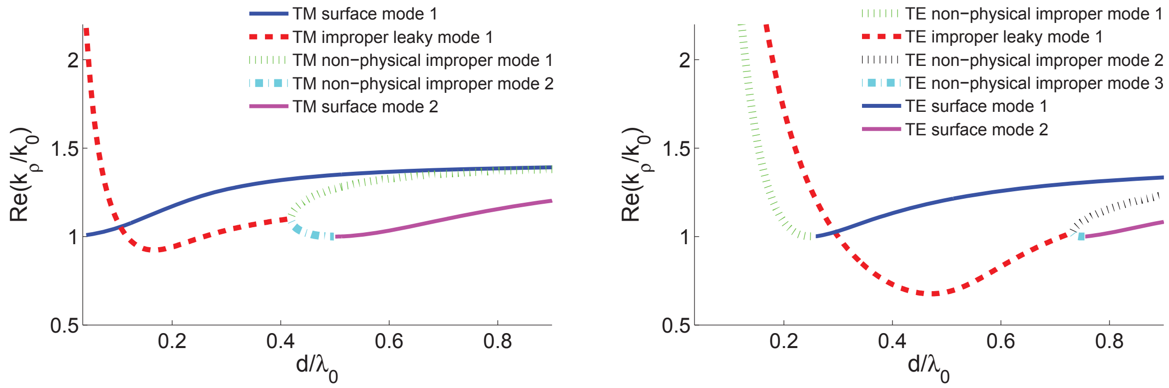


(a) (b) (c)

Figure 3: Evolution of normalized TE$_z$ modes in similar conditions as before, but in a different frequency range as compared to Fig. 2.



(a) (b)

Figure 4: Dispersion curves [Re(k_ρ/k_0)] for the TM$_z$ (a) and TE$_z$ (b) modes of an open grounded isotropic slab with $\varepsilon_r = 2$ and $\mu_r = 1$.

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