SCATTERING BY AN EDGED CYLINDER: NUMERICAL ANALYSIS BY THE YASUURA METHOD

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1. Introduction

We describe numerical techniques for solving the problem of scattering by an edged cylinder. Both the E-wave (electric field being parallel to the axis of the cylinder) and the H-wave (magnetic field parallel) incidences are considered. We apply the Yasuura method or the mode-matching method (MMM) [1-6] to the problem. A singular-smoothing procedure (SSP) [5,6] is employed to find a rapidly converging sequence of approximate solutions for the E-wave incidence. A combination of a smoothing procedure (SP) [3,4] and the SSP is utilized for the H-wave case. Numerical examples for a square cylinder show the effectiveness of the techniques.

2. Formulation

We consider the problem to find a wave-function $\Psi(P)$ satisfying the two-dimensional Helmholtz equation outside an edged contour C (Fig. 1), the radiation condition, and the Dirichlet or the Neumann boundary condition on the contour. The Dirichlet condition applies for the E-wave case:

$$\Psi(s) = f(s) = -E_z{}^{i}(s) \quad (0 \le s \le 1); \tag{1}$$

and the Neumann condition is for the H-wave case:

$$\partial_\nu \Psi(s) = g(s) = -\partial_\nu H_z{}^{i}(s) \quad (0 \le s \le 1) \tag{2}$$

where ∂_ν denotes the normal derivative.

3. Method of solution: E-wave case

Let us define an approximate solution by

$$\Psi_N(P) = \Sigma_{m=-N}^{N} A_m(N)\phi_m(P) \tag{3}$$

where $\phi_m(P)(m=0, \pm 1, \pm 2, \ldots)$ are modal functions given by

$$\phi_m(P) = H_m(k\rho)\exp(im\theta) \tag{4}$$

with H_m being the second type Hankel functions. Since the approximate solution satisfies the Helmholtz equation and the radiation condition, the A_m-coefficients should be determined so that the solution approximately satisfies the boundary condition.

Minimizing the norm+ $||\Psi_N - f||$, we can find an approximate solution. (a conventional MMM [1,2]) It is verified that the sequence of approximate solutions by this method converges to the true solution as the number of truncation tends to infinity. The sequence however converges often so slowly that we cannot find a precise solution.

The SSP accelerates the convergence of E-wave solutions. The algorithm of the MMM with the SP is as follows.

First we define the functions+

$$\psi_m(s) = \phi_m(s) - (1,\phi_m)\phi_\mu(s)/(1,\phi_\mu) \quad m \neq \mu \tag{5}$$

$$h(s) = f(s) - (1,f)\phi_\mu(s)/(1,\phi_\mu) \tag{6}$$

where $\phi_\mu(s)$ is an element which is not orthogonal to constants: $(1,\phi_\mu) \neq 0$. We make an approximate function for the h(s) in terms of a linear combination of the ψ_m-functions:

$$h_N(s) = \Sigma'^{N}_{m=-N} a_m(N)\psi_m(s) \tag{7}$$

+ $(\phi,\psi) = \int_0^1 \phi^*(s)\psi(s)\, ds$, $||\phi|| = (\phi,\phi)^{1/2}$.

where Σ' means a summation excluding $m=\mu$. The functions $\psi_m(s)$, $h(s)$, and $h_N(s)$ are orthogonal to constants: $(1,\psi_m)=0$, etc. It is worth noting that the approximation "$h(s)\fallingdotseq h_N(s)$" is equivalent to "$f(s)\fallingdotseq\Psi_N(s)$ under the constraint that $(1,f)=(1,\Psi_N)$"; the relationship between the a_m- and the A_m-coefficients is that

$$A_\mu(N) = -[\Sigma'^{N}_{n=-N}(1,\phi_n)a_n(N) + (1,f)]/(1,\phi_\mu)$$
$$A_m(N) = a_m(N) \quad (m \neq \mu) \tag{8}$$

Next we define an operator of the SSP by

$$\mathbb{H}\psi(s) = \int_0^1 H(s,t)\psi(t)\,dt, \quad H(s,t) = u(s-t) - u(\xi-t) \tag{9}$$

The function $\mathbb{H}\psi(s)$ is an indefinite integral of $\psi(s)$ having a zero at $s=\xi$:

$$d\mathbb{H}\psi(s)/ds = \psi(s), \quad \mathbb{H}\psi(\xi) = 0 \tag{10}$$

The operator $\mathbb{H}$ is needed because of the singularity due to the edge point.

We determine the a_m-coefficients so that the norm

$$\Omega_N = ||\mathbb{H}(h_N-h)/(s-\xi)||^2 \tag{11}$$

becomes minimum. And the A_m-coefficients are found by (8). We can prove that the sequence obtained by the method above converges to the true solution uniformly in any closed subdomain in the exterior infinite domain.

In actual numerical computations we should devide the interval [0,1] into $J=2(2N+1)$ subsections and discretize the least-squares problem. A recommended way to solve the discretized problem is the QR algorithm.[7] This is the reason why we have introduced the ψ_m-functions: we cannot employ the algorithm so long as we try to find the A_m-coefficients directly.

4. Method of solution: H-wave case

In this polarization, the SP accelerates the convergence to a certain degree. However a combination of the SP and the SSP is superior to the SP because of the rapid convergence and a wide range of applicability.

The ψ_m-functions, etc. are defined by

$$\psi_m(s) = \partial_\nu\phi_m(s) - (1,\partial_\nu\phi_m)\partial_\nu\phi_\lambda(s)/(1,\partial_\nu\phi_\lambda) \quad m \neq \lambda \tag{12}$$
$$h(s) = g(s) - (1,g)\partial_\nu\phi_\lambda(s)/(1,\partial_\nu\phi_\lambda) \tag{13}$$
$$h_N(s) = \Sigma'^{N}_{m=-N}\, a_m(N)\psi_m(s) \tag{14}$$

where λ is an integer such that $(1,\partial_\nu\phi_\lambda)\neq 0$, and Σ' denotes a summation excluding $m=\lambda$. The relationship between the a_m- and the A_m-coefficients is given by (8).

We determine the a_m-coefficients so as to minimize the norm

$$\Omega_N = ||\mathbb{H}\mathbb{K}(h_N-h)/(s-\xi)||^2 \tag{15}$$

Here, $\mathbb{K}$ denotes an operator of the SP which is defined by

$$\mathbb{K}\psi(s) = \int_0^1 K(s,t)\psi(t)\,dt, \quad K(s,t) = u(s-t) - (s-t) - 1/2 \tag{16}$$

This operator transforms a function into an indefinite integral of a component of the original function being orthogonal to constants:

$$d\mathbb{K}\psi(s)/ds = \psi_\perp(s) = \psi(s) - (1,\psi) \tag{17}$$

And the $\mathbb{K}\psi(s)$ again is orthogonal to constants: $(1,\mathbb{K}\psi)=0$.

We can prove the convergence theorem for this case. Discretization and numerical analysis should be made in the same way as in Section 3.

5. Application to scattering by a square cylinder

We analyse the problem of plane-wave scattering by a perfectly conducting square cylinder shown in Figure 2.

The cylinder has geometrical symmetry on rotation through about the z-axis, and hence the polyphase wave functions[8] can be employed to save much numerical computation.

Figures 3 and 4 show the decrement of error on the optical theorem (i.e., energy error). We can easily find precise solutions with errors less than 0.1% for a wide range in the resonance region. Figure 5 illustrates an example of scattering pattern. The backscattering cross section for the H-wave incidence is shown in Figure 6 as function of the wavenumber.

6. Concluding remarks

We described a method of solution for the problem of scattering by an edged cylinder. The method was applied to the problem of a square cylinder and the numerical results demonstrate the effectiveness of the method.

It should be noted that the number of divisions was decided by $J=2(2N+1)$. We can verify numerically that the choice is resonable by singular value analysis. [9]

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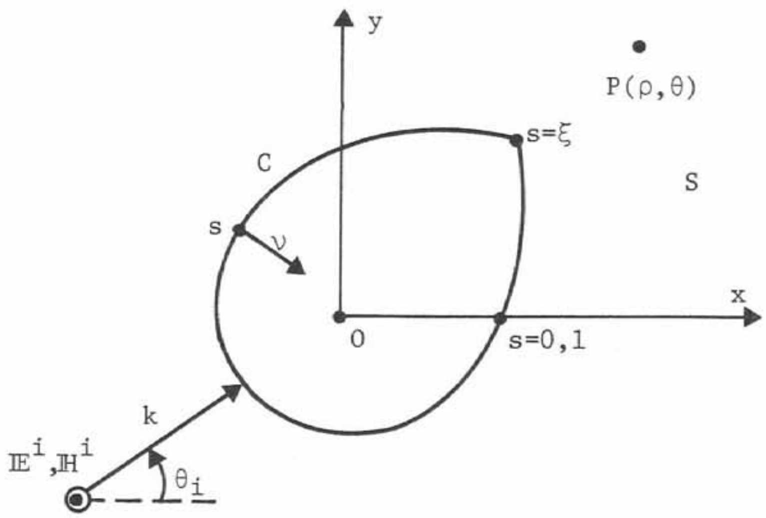


Fig. 1. The cross section of an edged cylinder. The length of C is normalized to be unity. The edge point locates at $s=\xi$.

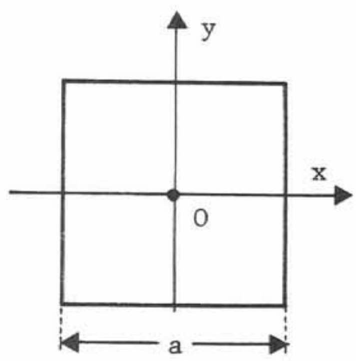
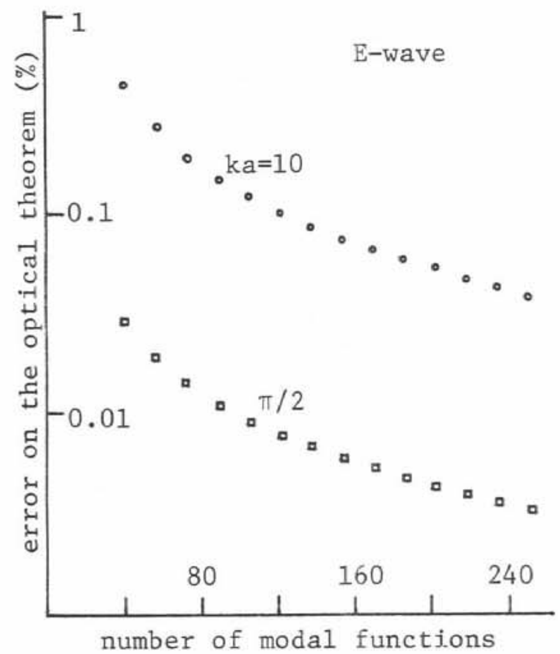
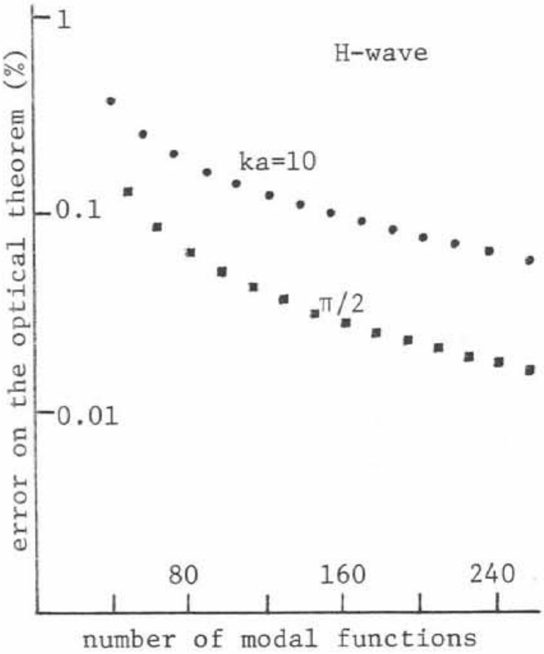


Fig. 2. The cross section of a square cylinder for sample calculation.





Figs. 3(left) and 4(right). Decrement of the error on the optical theorem as the number of truncation increases. $\theta_i = 45°$.

Fig. 5. A sample of scattering pattern. $ka = 4\pi$, $\theta_i = 45°$. ⟶

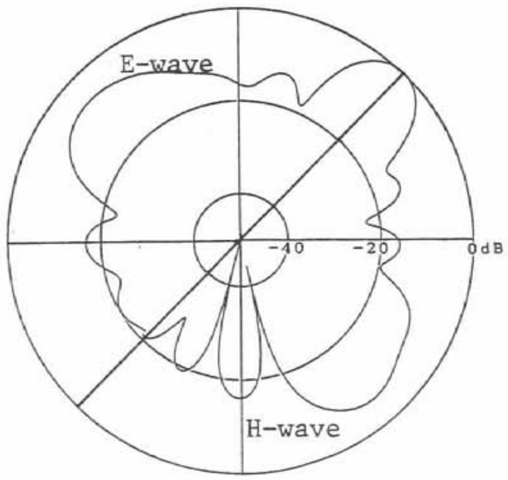


Fig. 6. Backscattering cross section for some incidence angles. H-wave.

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