Proceedings of ISAP'04, Sendai, JAPAN

A New Least Square Based Method for the Solution of Parabolic Equation

Homayoon Oraizi[1] and Shahram Hosseinzadeh[2]

[1]Professor and [2]Ph.D. Student
Department of Electrical Engineering, Iran University of Science and Technology
Narmak, 16844 Tehran, Iran, Phone +98-28-7808022, h oraizi@iust.ac.ir

Abstract— **In this paper a new method for the solution of parabolic equation in troposphere will be presented.Electromagnetic field will be expanded by proper expansion functions. These expansion functions satisfy the parabolic equation in homogeneous media and by means of least square method, the expansion coefficients will be derived for initial and boundary conditions. The least square functionals satisfy the initial coefficient directly and by means of Lagrange multipliers the boundary condition will be exerted. This method is more reliable than the split step method and can be applied over rough impedance boundary in a simple manner. In comparison with the finite difference method, the proposed method is very fast.**

I. INTRODUCTION

Recent advances in planning and implementation of wireless radio communication systems and the sensivity of these systems to interference and multipath phenomena, require efficient usage of frequency spectrum. Therefore, before planning these systems, their internal and mutual interference effects on other radio systems and the effect of multipath on the radio system must be accurately determined. Field strength due to each system must be calculated. Therefore, one of the radio wave propagation models must be used. These models fall into empirical, semi-empirical and theoretical models. Empirical and semi-empirical models are based on field trials in various geographical areas.Theorotical models are based on approximate solution of Maxwell equations.[1]

In addition to surface roughness, a suitable model must take the effect of refractive index of troposphere into account. Advection, subsidence and frontal produce stratification of troposphere and change the radius of ray. Time variations of refractive index create large variations of amplitudes of field. Atmospheric duct decreases the propagation loss and causes inter-service interference[2]. The majority of empirical and semi-empirical models neglect the effect of refractive index on the field strength. Some of these models only consider the effect of simple refractive index profiles(linear or exponential). Therefore, the error of field strength computation by these models for large variation of refractive index profile is high and their reliability is very low[1].

Theoretical models are divided into models based on ray theory and modal theory. In the ray theory based models, at first the propagating ray is identified (for example direct ray, reflected ray , diffracted ray and ...) and then by tracing each ray the field strange is calculated. Qualitative prediction can be made by this model. However, near caustics (for example atmospheric duct) this model fails and loses its accuracy[3]. Also , mistake in classification of rays causes error in prediction. For example, neglecting lateral waves in forest environment for transmitters and receivers located inside forests, causes large errors in the prediction of radio wave propagation[4]. In the modal based methods, by modal expansion of fields, and considering their orthogonality, the amplitude of each mode is calculated. Computation of modes in media with irregular refractive index is a cumbersome task. Also the convergence rate of modal method is very slow. Therefore, the computation cost is excessive. In practice a method composed of the two aforementioned methods is used.

Recently, marching algorithm was used for the analysis of radio wave propagation problem[11-13]. In comparison with other scattering problems, the domain of radio wave propagation problem is very large. Therefore, their numerical computation requires excessive computer memory. In the marching algorithm , at first the problem is solved in smaller sub-domains, required data is saved and excess data is discarded. This method makes possible the optimum use of memory resources.The majority of marching algorithms are devised for the parabolic approximation of wave equation whereby the backward propagated field is neglected and parabolic equation(PE) for forward field is extracted.

Finite difference(FD) and split step(SS) method have been used for the solution of PEM in the troposphere[3]. In the finite difference method, the derivatives of parabolic equation are replaced by appropriate difference equations and the partial differential equation is changed into a matrix equation. In the split step method, at first the effect of refractive index is separated from the effect of surfaces. The effect of refractive index is implemented by proper phase shift of field in homogeneous media.

In this paper, based on the least square method a marching algorithm for modeling of radio wave propagation in the troposphere will be presented. This algorithm is quite reliable for the radio wave propagation problem in the troposphere with various profiles of refractive index.

It the second section of the paper, the problem is formulated and in the third and fourth sections of paper, based on formulas presented in the second section, least square functionals will be constructed. By minimization of these of this functionals, the field in troposphere will be calculated. Finally the results of this paper are compared with those of other methods.

II. PROBLEM FORMULATION

Radio wave propagation modeling requires the solution of wave equation with proper boundary conditions.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} + k^2 n^2 \varphi = 0 \tag{1}$$

where the field is considered y independent, k and n are propagation constant and refractive index respectively, φ is either E_y or H_y depending on polarization. We define u as follows[13].

$$u(x, z) = \varphi(x, z) e^{-jkx} \tag{2}$$

By substituting (2) in (1) and some approximation a partial differential equation for u can be extracted.

$$\frac{\partial^2 u}{\partial z^2} + 2jk\frac{\partial u}{\partial x} + k^2 \left(n^2 - 1\right) u = 0 \tag{3}$$

Conductivity and dielectric constant of surface determine the appropriate boundary conditions. For a conductive earth, Dirichlet($u(x, 0)$) and or Neumann boundary condition($\frac{\partial u(x,0)}{\partial z}$), depending on polarization must be satisfied. However, conductivity and dielectric index of earth have finite values depending on geographical region. Therefore, Neumann and Dirichlet boundary conditions cause errors in the propagation modeling(especially at low frequencies). For this

ISBN: 4-88552-207-2 C3055©IEICE

case, the field inside the earth must be calculated and the continuity condition of field components must be satisfied. This increases of computations.
For decreasing computations, approximate impedance boundary condition has been proposed whereby only the field inside the domain of interest is calculated. Several impedance boundary conditions have been proposed. Their accuracy depend on their complexity.
Leontovich impedance boundary condition is the simplest in which the tangential component of electrical field is considered proportional to the tangential component of magnetic field.

$$\frac{\partial u}{\partial z} + \alpha u = 0 \tag{4}$$

Proportionally factor is computed by considering the plane wave reflection from earth. Although this condition is a simple, it has good accuracy and has been used in several of radio wave propagation problems. For example at for radio wave propagation over random surfaces (for example sea surface) , by considering an effective height of surface, an appropriate proportionally factor is computed and the problem of wave propagation over rough surface reduces to wave propagation over lossy earth.
In this paper, similar to the split step method, the effect of refractive index and boundary condition are considered separately. At first the problem for the homogeneous media($n = 1$) is solved.

$$\begin{gathered} \frac{\partial^2 u}{\partial z^2} + 2jk\frac{\partial u}{\partial x} = 0 \\ u(0,z) = f(z) \\ \frac{\partial u(x,0)}{\partial z} + \alpha u(x,0) = 0 \end{gathered} \tag{5}$$

Then by multiplying the computed field by $e^{-jk\frac{n^2-1}{2}}$, the effect of troposphere is applied.
We truncate the upper height of problem domain at fixed finite height($z = h$). For u we consider the following solution:

$$u(x,z) = \sum_{m=-\infty}^{\infty} a_m^* e^{-\frac{1}{j2k}\left(\frac{2m\pi}{h}\right)^2 x} e^{\frac{2m\pi}{h} z} \tag{6}$$

This equation satisfies the parabolic equation in homogeneous media. In order to satisfy initial and boundary conditions the coefficients a_m^* must be determined appropriately.

$$\sum_{m=-\infty}^{\infty} a_m^* e^{\frac{2m\pi}{h} z} = f(z) \tag{7}$$

$$\sum_{m=-\infty}^{\infty} a_m^* \left(\alpha + \frac{j2\pi m}{h}\right) e^{-\frac{1}{j2k}\left(\frac{2m\pi}{h}\right)^2 x} = 0 \tag{8}$$

III. Construction of Least square Functional

With due attention to satisfaction of the parabolic equation by expansion functions, the least square functional must be constructed in such a manner to satisfy initial and boundary conditions. Similar to other solution methods of parabolic equation , its solution in a finite interval is considered. The least square functional is constructed directly for the initial conditions and by means of language multiplier for boundary conditions.
By defining the inner product and norm as

$$\begin{gathered} \langle v,w\rangle = \frac{1}{h}\int_0^h vw^* dz \\ \|v\|^2 = \langle v,v\rangle \end{gathered} \tag{9}$$

The least square functional can be written as follows

$$F = \|u(0,z) - f(z)\|^2 + Re\sum_{n=0}^{N-1}\lambda_n\left(\frac{\partial u}{\partial z} + \alpha u\right)(x_n,0) \tag{10}$$

where $(x_n, 0)$ shows the sampling point on the earth surface ,λ_n are Lagrange multipliers and Re stands for real part of function.
By some mathematical manipulation, the functional can be written as

$$\begin{aligned} F = \sum_{m=-\infty}^{+\infty} |a_m|^2 + \|f(z)\|^2 - \sum_{m=-\infty}^{+\infty} a_m^* f_m - \sum_{m=-\infty}^{+\infty} f_m^* a_m \\ + Re\sum_{n=0}^{N-1}\lambda_n \sum_{m=-\infty}^{+\infty} a_m^* \left(\alpha + \frac{j2\pi m}{h}\right) e^{-\frac{1}{j2k}\left(\frac{2m\pi}{h}\right)^2 x} \end{aligned} \tag{11}$$

where

$$f_m = \left\langle e^{j\frac{2m\pi}{h}z}, f(z)\right\rangle \tag{12}$$

IV. Functional Matrix Formulation

Matrix formulation of the functional facilitates its mathematical manipulation. At first, we truncate the infinite summations

$$u(x,z) = \sum_{m=-M}^{M-1} a_m^* e^{-\frac{1}{j2k}\left(\frac{2m\pi}{h}\right)^2 x} e^{\frac{2m\pi}{h} z} \tag{13}$$

Therefore, we can approximate the functional as follow

$$\begin{aligned} F = \sum_{m=-M}^{M-1} |a_m|^2 - \sum_{m=-M}^{M-1} a_m^* f_m - \sum_{m=-M}^{M-1} f_m^* a_m + \|f(z)\|^2 \\ + Re\sum_{n=0}^{N-1}\lambda_n \sum_{m=-M}^{M-1} a_m^* \left(\alpha + \frac{j2\pi m}{h}\right) e^{-\frac{1}{j2k}\left(\frac{2m\pi}{h}\right)^2 x} \end{aligned}$$

We define following matrices

$$\underline{a} = [a_{-M}, a_{-M+1}, ..., a_{M-1}]^T \tag{14}$$

$$\underline{f} = [f_{-M}, f_{-M+1}, ..., f_{M-1}]^T \tag{15}$$

$$\underline{s_n} = [s_{n,-M}, s_{n,-M+1}, ..., s_{n,M-1}]^T \tag{16}$$

$$\underline{\lambda} = [\lambda_0, \lambda_1, ..., \lambda_n]^T \tag{17}$$

where $0 \le n \le N-1$ and

$$\left(\alpha + \frac{j2\pi m}{h}\right) e^{-\frac{1}{j2k}\left(\frac{2m\pi}{h}\right)^2 x} \tag{18}$$

Also matrix $\underline{\underline{S}}$ is defined as

$$\underline{\underline{S}} = [s_0, s_1, ..., s_{N-1}] \tag{19}$$

Therefore we can write the functional in a closed form

$$F = \underline{a}^H\underline{a} - \underline{f}^H\underline{a} - \underline{a}^H\underline{f} + Re\left(\underline{a}^H\underline{\underline{S}}\underline{\lambda}\right) + \|f(z)\|^2 \tag{20}$$

V. Functional Minimization

Minimization of functional requires that its gradient become zero[14].

$$\nabla F = \frac{\partial F}{\partial \underline{a}^H} = \underline{a} - \underline{f} + \underline{\underline{S}}\underline{\lambda} = 0 \tag{21}$$

Therefore $\underline{a}$ can be evaluated as follow

$$\underline{a} = \underline{f} - \underline{\underline{S}}\underline{\lambda} \tag{22}$$

For finding the Lagrange multipliers, we exert the boundary condition.

$$\underline{a}^H\underline{\underline{S}} = 0 \tag{23}$$

Therefore Lagrange multipliers can be found as follows

$$\underline{\lambda} = \left(\underline{\underline{S}}^H\underline{\underline{S}}\right)^{-1}\left(\underline{\underline{S}}^H\underline{f}\right) \tag{24}$$

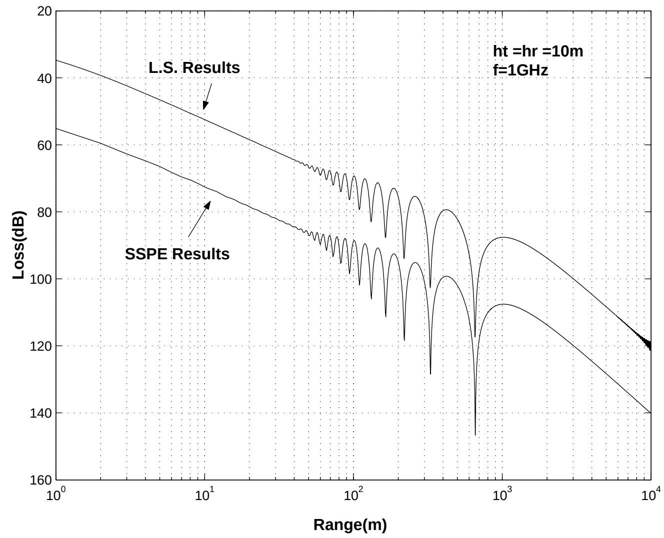

Fig. 1. Comparison of results of LS algorithm with Split Step method for horizontal polarization. split step results have 20dB offset for clarity

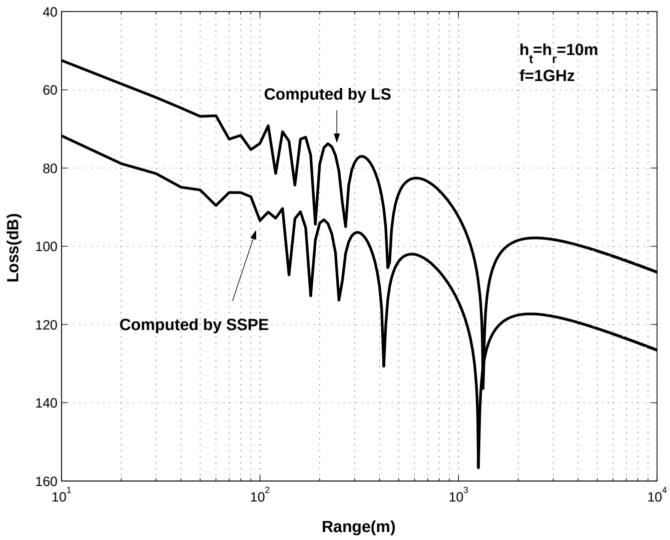

Fig. 2. Comparison of results of LS algorithm with Split Step method for vertical polarization. split step results have 20dB offset for clarity

VI. Numerical Results

To investigate the efficacy of proposed algorithm, we apply it to several boundary conditions and compare the results with other methods. Also we investigate the effect of number of sampling points on the accuracy of results. FFT algorithm is used for calculating $\underline{f}$ and $\underline{a}$.

Fig.1 compares the propagation loss for horizontal polarization computed by the proposed algorithm with propagation loss computed with split step method. It is seen that the results exactly coincide. Fig.2 shows similar results for vertical polarization. Fig.3 and Fig.4 compare the propagation loss over earth with real α by results of finite difference method. It is seen that the results agree for both α

Fig.6 and Fig.7 show the propagation loss in the presence of terrain up-ward and down-ward steps, as shown in the Fig.5. It is seen that the results of least square method for both up-ward and down-ward steps agree with the results of split step method.

Fig.8 and Fig.9 show the effect of sampling number on the propagation loss. It is observed that the increase of the sampling number reduces the error in propagation loss. However, excessive increase of the sampling number causes ill-conditioning of $\left(\underline{\underline{S}}^H\underline{\underline{S}}\right)$ at (25). This ill-conditioning causes error of the calculation of

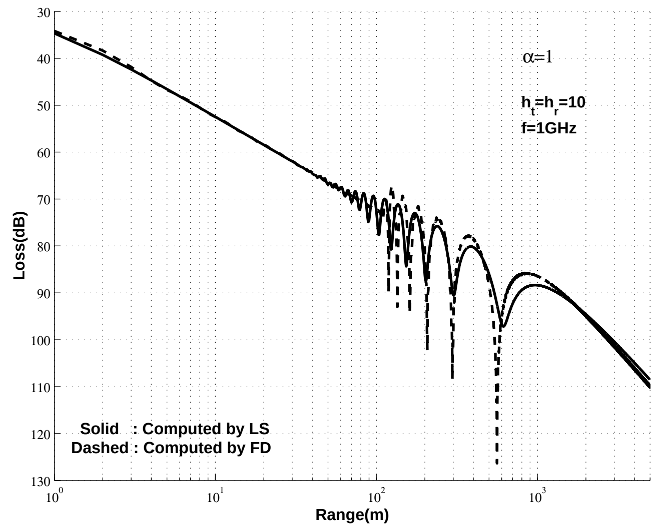

Fig. 3. Comparison of results of LS algorithm with Finite Difference method for $\alpha = 1$.

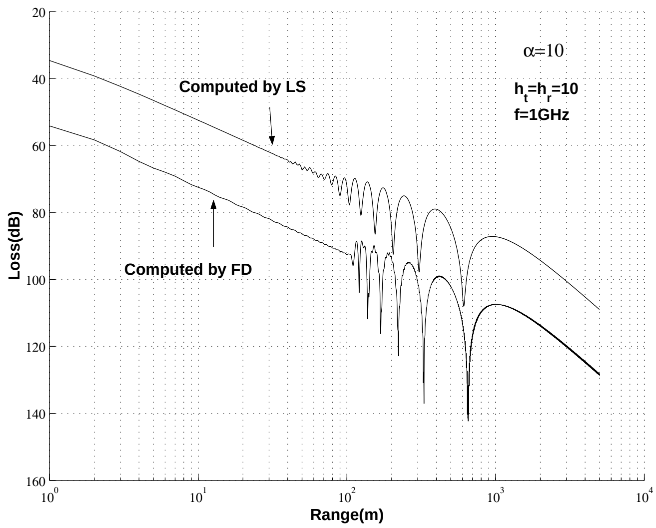

Fig. 4. Comparison of results of LS algorithm with Finite Difference method for $\alpha = 10$.Finite Difference results have 20dB offset for clarity

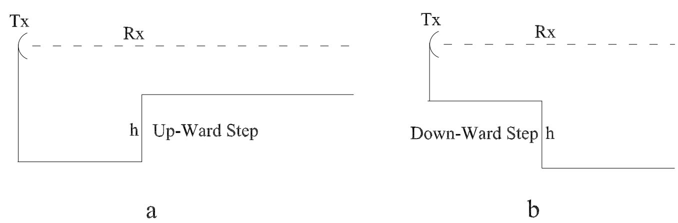

Fig. 5. Geometry of (a)Up-ward and (b) Down-ward Steps.

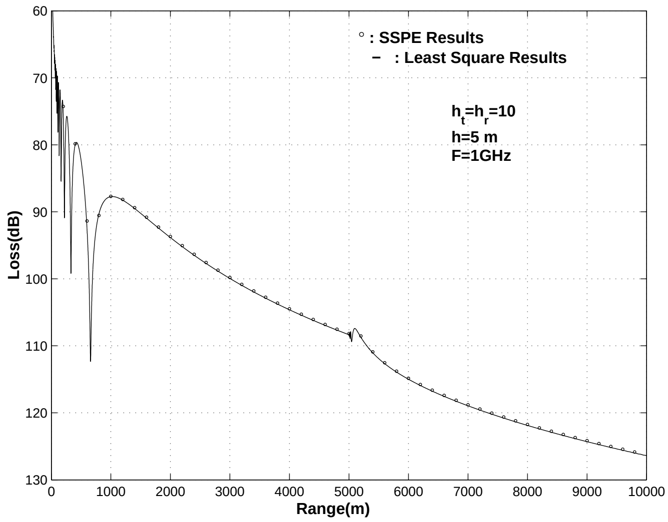

Fig. 6. Propagation loss in the presence of up-ward step

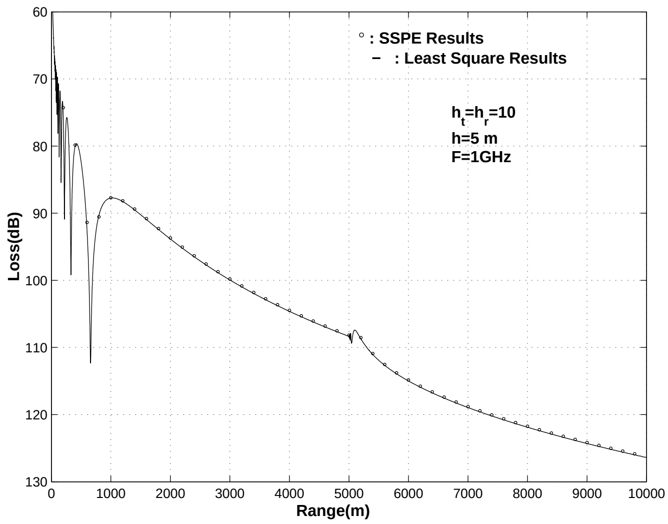

Fig. 7. Propagation loss in the presence of down-ward step

propagation loss and instability of the algorithm. For example fig.9 shows that ill-conditioning produces minus propagation loss which is not acceptable.

Further investigations show that maximum sampling number depends on the number of terms in (14). By increasing the terms, maximum allowable sampling number also increases.
Also author's simulation experiments show that maximum and minimum number of sampling points depend on height and range resolution. Increasing range resolution increases the required minimum number of sampling points in order to achieve the acceptable accuracy. Decreasing height resolution increases the maximum allowable sampling points in order to prevent ill-conditioning.
Therefore, the computation time depends on the range and height resolution. The computation time of algorithm is larger than split step computation time and less than finite difference method.

VII. CONCLUSION

In this paper a new method for the solution of parabolic equation in the troposphere over lossy earth is presented. It is seen that the results of Least Square Method exactly agree with those of other methods for the solution of parabolic equation. Also FFT algorithm can be used for the reduction of computation time. Various boundary conditions can be applied with Lagrange multipliers. Therefore, the algorithm is flexible. However, for preventing ill-conditioning of the algorithm and increasing its accuracy, the number of sampling points must be chosen accurately.

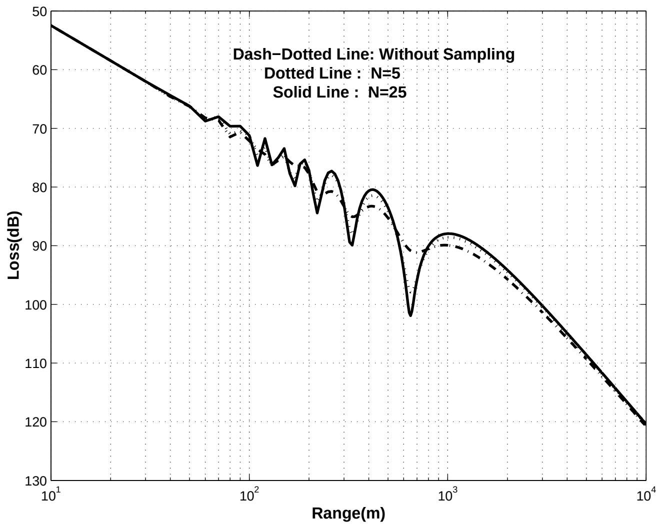

Fig. 8. The effect of number of sampling points on the accuracy of propagation loss.

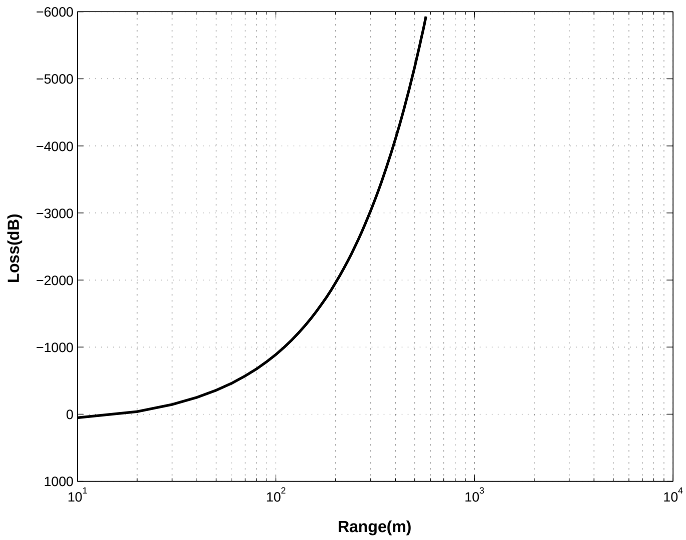

Fig. 9. The effect of excess number of sampling points on the ill-conditioning of algorithm

REFERENCES

[1] N. Blaunstein, Radiwave Propagatin in Cellulat Networks, Artech House Mobile Comunication Library,1999
[2] H. Oraizi and S. Hosseinzadeh, " The Effect of Atmospheric Duct on Modern OFDM-Based Digital Broadcasting System" Proceeding of 33^{rd} European Microwave Conference, EuMC2003,PP. 747-750,Munich-Germany
[3] P. L. Slingby," Modelling Tropospheric Ducting Effect on VHF/UHF Propagation" ,IEEE Trans. Broadcasting, Vol.37,No.2,June 1991
[4] T. Tamir,"On Radio-Wave Propagation in Forest Environment" IEEE Transaction on Antennas and Propagation,Vol.15,No.6,Nov.1967,PP. 806-807
[5] R. Janaswammy," Radio Wave Propagation over Non-constant Immitance Plane" , Radio Sci.,Vol. 36, No.3, PP. 387-405,May-June 2001
[6] A. A. Zaporzhests and M. F. Levy,Final Report on Hybrid Field/Current Marching Techniques for Path Loss Calculations, RCRU-RA Report D52-5,2000
[7] M. F. Levy, Paraboloic Equation Method for Electromagnetic Wave Propagation, IEE Press 2000
[8] S. Haykin, Adaptive Filter Theory, Prentice Hall International Publication, 1991