Investigation of Application of Higher-Order Elements in FEM

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***Abstract*— Application of arbitrary-degree elements is a feature that is not yet practically deployed in commercial or scientific FEM software. The present paper provides reasons supporting the idea of developing arbitrary-degree element software, which would have considerable benefits to FE analysis. Here, some simple FEM modal problems are analyzed as examples to illustrate the proposed idea.**

***Index Terms*— arbitrary-degree elements, FEM.**

I. INTRODUCTION

THE benefit of applying high-order elements in FEM has been known for a long time[1]. Some generalizing attempts have been carried out to deploy higher-order elements in FEM analysis[4]. However, little has been attempted on arbitrary-degree, higher-order elements in FEM analysis. In general, the application of arbitrary-degree high-order elements in FEM solutions of field problems requires two fundamental criteria to be met and they are namely:

1) An arbitrary-degree mesh/mesh-generator.
2) The availability of arbitrary-degree fundamental element matrices, which have been preferably evaluated using analytical approach.

In addition, a generalized formulation relating various types and classes of element matrices to possible fundamental matrices must be derived. In this paper, a general formulation on arbitrary-degree, higher-order element for analysis has been presented. As a demonstration, the modal solutions of both a homogeneous and nonhomogeneous partially loaded waveguide are evaluated using various degrees of interpolation. Finally the results are compared to those generated using analytical formulation and to those generated using conventional low order elements.

II. ARBITRARY-DEGREE INTERPOLATION

As depicted in Fig. 1, in an arbitrarily sized rectangular element, an arbitrary number of fractions along each dimension is assumed. Consequently $(m+1)$ by $(n+1)$ interpolation nodes would be needed in the element. A two-dimensional extension of Lagrangian interpolation of an arbitrary function $\Phi(x,y)$ over the element is presented in equation (1):

$$\Phi = \sum_{i=1}^{(m+1)\times(n+1)} \phi_i \alpha_i(x,y) : x_1 < x < x_2, y_1 < y < y_2, \tag{1}$$

where i$\in [1,(m+1)(n+1)]$ is the node index in the element and the functions $\alpha_i(x,y)$ are the required shape functions that meet Lagrange interpolation properties. This is the key to all proceeding steps in the present paper. An alternate way of representing the nodes is to use a vector-formed two-dimensional index as (p,q). When the local coordinates are used and equidistant interpolation nodes are adopted, $\alpha_{(p,q)}(\xi,\zeta)$ can assume as

$$\alpha_{(p,q)}(\xi,\zeta) = \frac{m^m \prod_{s=0}^{p-1}(\xi - s/m)\prod_{s=p+1}^{m}(\xi - s/m)}{p!(m-p)!(-1)^{m-p}} \times \frac{n^n \prod_{s=0}^{q-1}(\zeta - s/n)\prod_{s=q+1}^{m}(\zeta - s/n)}{q!(n-q)!(-1)^{n-q}}, \tag{2}$$

Equation (2) can be directly used to calculate the necessary element matrices for finite element calculations. In this work, a symbolic math calculation code written in *"Waterloo Maple"* are used to obtain the fundamental matrices within accurate symbolic regime, i.e. they contain exact fractional numbers. Although the computation of such fundamental elements may require considerable amounts of CPU time, it is not a matter of concern since these results are stored in a fixed database once and forever. Each elements matrix can be calculated as a weighted combination of the mentioned fundamental elements' matrices eventually.

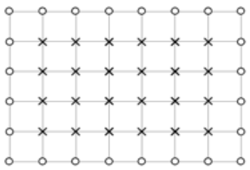

Fig. 1. Subdividing a rectangular element with fraction numbers m = 7 and n = 5

III. ERROR ESTIMATES

For modal FEM problems and problems with no distributed excitation within problem domain, the actual location of interpolation points plays little role in FEM interpolation error. Unlike direct approximation of known functions, FEM analysis leads to a polynomial that meets the conditions for the minimization of a variational form. Since the degree of the polynomial is constant and known, the final resulting polynomial is unique and is not affected by the choice of interpolation nodes. Thus, for the study of asymptotic behavior of error, it is sufficient to use the error bound estimate for a two-variable Lagrange interpolation, given[2] as

$$|E_{m,n}(x,y)| \le \frac{Max|\frac{\partial \Psi^{(m+n+2)}}{\partial x^{m+1}\partial y^{n+1}}|}{(m+1)!(n+1)!}, \tag{3}$$

Unless interpolation of some spurious function(s) like that of Runge's counter example is intended, the interpolation

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would converge to the true result as both $(m+1)$ and $(n+1)$ keep increasing simultaneously. Situations like that of Runge's counter example do not arise in many FEM problems unless solution of problems with distributed excitation (eg. Poisson's Equation) is considered.

IV. FEM Formulation

For demonstration, a rectangular waveguide, which is filled homogeneously with one single isotropic material is considered first. followed by the same waveguide filled with two different layers of dielectrics. The vector variational forms for the nonhomogeneous case are borrowed from [1].

A. The Homogeneous Case

The FEM formulation for homogeneous waveguide problems can be stated as follows:

$$\mathcal{A}\ [\psi] = k_t^2\ \mathcal{B}\ [\psi], \tag{4}$$

where

$$\mathcal{A}^e = \int_{\Omega_e} (\frac{\partial \alpha_i^e(x,y)}{\partial x}\frac{\partial \alpha_j^e(x,y)}{\partial x} + \frac{\partial \alpha_i^e(x,y)}{\partial y}\frac{\partial \alpha_j^e(x,y)}{\partial y})\ dx\ dy, \tag{5}$$

$$\mathcal{B}^e = \int_{\Omega_e} \alpha_i^e(x,y)\alpha_j^e(x,y)\ dx\ dy, \tag{6}$$

the superscript index "e" in the mentioned equations indicates that these are element matrices. Thus global coefficients matrix must be appropriately assembled. Indeed, before solving the obtained matrix eigenvalue problem, appropriate boundary conditions should be applied whenever needed. Note that ψ stands for a vector of nodal values of either E_z or H_z depending on the choice of TM^z or TE^z modes of the waveguide.

B. The Nonhomogeneous Case

For the nonhomogeneous case, the formulation is slightly more complex. The existing modes are no longer expressible as TE^z or TM^z fields ($\hat{z}$ is the direction of propagation). A straight forward FEM formulation for the problem follows in equation (7):

$$\frac{1}{\epsilon_r^e\mu_r^e-\delta^2} \begin{bmatrix} \epsilon_r^e\mathcal{A}_0 & \delta\mathcal{C}_0 \\ \delta\mathcal{C}_0^T & \mu_r^e\mathcal{A}_0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_z \\ Z_0\mathcal{H}_z \end{bmatrix} = k_0^2 \begin{bmatrix} \epsilon_r^e\mathcal{B}_0 & 0 \\ 0 & \mu_r^e\mathcal{B}_0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_z \\ Z_0\mathcal{H}_z \end{bmatrix}, \tag{7}$$

$$\delta \triangleq \frac{k_z}{k_0}, \tag{8}$$

$$\mathcal{C}_{0ij}^e = \int_{\Omega_e} (\frac{\partial \alpha_i^e(x,y)}{\partial x}\frac{\partial \alpha_j^e(x,y)}{\partial y} - \frac{\partial \alpha_i^e(x,y)}{\partial y}\frac{\partial \alpha_j^e(x,y)}{\partial x})\ dx\ dy, \tag{9}$$

$\mathcal{A}_{0ij}^e$ is equivalent to $\mathcal{A}_{ij}^e$ in equation (5), $\mathcal{E}_z$ and $\mathcal{H}_z$ represent vectors containing nodal values of z-component electric and magnetic field phasors across the problem domain. Similarly $\mathcal{B}_{0ij}^e$ appears to be equivalent to $\mathcal{B}_{ij}^e$ of equation (6). Equation (9) provides the definition for $\mathcal{C}_{0ij}^e$. The global matrices in equation (7) must be properly assembled from their element based equivalents. It is noted that the form of this formulation does not directly comply with that of a generalized eigenvalue/vector problems. The trick, however, is to assume different values of δ and to calculate the corresponding eigenvalues/vectors for each. The range in which δ should be varied, is $[0,1)\bigcup(1,\epsilon_{rMax})$, where ϵ_{rMax} is the maximum of dielectric constant ϵ_r across waveguide's cross section. Note that at $\delta=1$ and $\delta=\epsilon_{rMax}$, our problem becomes singular and thus these values should be avoided.

V. Numerical Results

For demonstration, a rectangular waveguide with a cross section of $(a \times b)\ m^2$ is considered. The cross section is orthogonally divided into $M \times N$ rectangular elements (M fractions along $\hat{x}$ and N fractions along $\hat{y}$). Each element is internally subdivided into $m \times n$ subregions resulting in $(m+1)\times(n+1)$ nodes each. For both the nonhomogeneous and homogenous examples stated here, the dimensions of the waveguide are assumed to be $a = 0.03m$ and $b = 0.02m$. To obtain equal mesh and node density along both directions (ie. $\hat{x}$ and $\hat{y}$), $\frac{M}{N} = \frac{a}{b} = \frac{3}{2} = \frac{3k}{2k}$ and $m = n = l$ is assumed. In Figs. (2), (3) and (4), the horizontal axis represents the parameter k while the vertical axis stands for l. There would be $(M\times m+1)\times(N\times n+1) = (3k\times l+1)(2k\times l+1)$ nodes for our problem. The number of elements would also follow the function $3k \times 2k$.

In general, k increases the number of elements and the total number of nodes would increase consequently. At the same time, l would directly increase the degree of interpolation used, which leads to an increase in the number of nodes, while the number of elements used is kept constant. Figs. (2), (3) and (4) provide a contour-plot of equl-accurate choices of k and l. Another contour that has been added to the mentioned figures is the plot of equal CPU time contours. The first set of contours is obtained by drawing a contour plot of the following error function:

$$\mathbf{E} = log_2|f_c - \tilde{f}_c|, \tag{10}$$

where f_c and $\tilde{f}_c$ stand for the values of numerically and analytically calculated cut-off frequency of a particular mode. Only TE^z mode(s) results are provided here, but it must be stated that TM^z mode(s) results also obey similar conditions in terms of error and accuracy. The CPU time contours are drawn as a decimal base logarithm of consumed CPU time(s) in seconds.

From the mentioned graphs of Figs. (2), (3) and (4), two major observations can be made. To reach maximum accuracy (or minimum error), it is far more efficiently achieved by moving along l rather than k. The other observation is that (higher modes) maximum computation accuracy can be achieved at higher CPU time costs. For the nonhomogeneous case, error and computation-time curves do demonstrate similar properties.

The nonhomogeneous example here deals with the same waveguide that has been half loaded by a layer of dielectric with $\epsilon_r = 4$ located in $0 < x < a;\ 0 < y < \frac{b}{2}$. Closed form equations for partially loaded waveguide's cut-off frequencies are available in most advanced EM texts. These are usually called LSE and LSM modes of partially loaded waveguides.

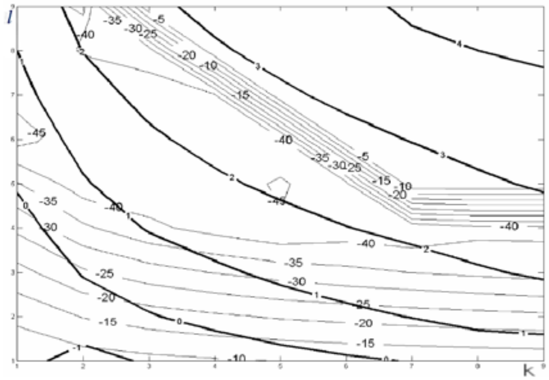

Fig. 2. Error (thin) and Computation Time (thick) contours for the 1^{st} TE^z mode eigen value vs k and l

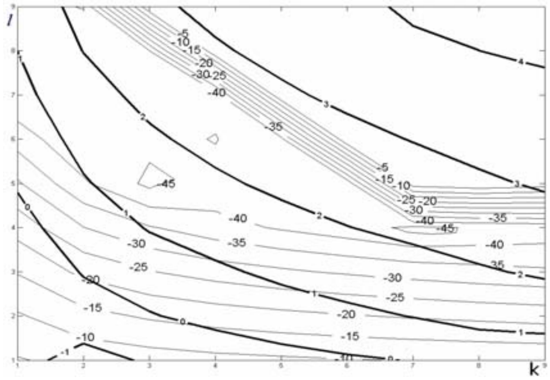

Fig. 3. Error (thin) and Computation Time (thick) contours for the 3^{rd} TE^z mode eigen value vs k and l

Table (I) and Table (II) provide the comparing of analytical and numerical cut-off frequencies for both homogeneous and nonhomogeneous cases using *4-node elements* and *64-node elements*. In order to keep the computational complexity approximately at the same level, the total number of nodes is kept constant by varying the number of elements at the same time. As observed in Tables (I) and (II), high-order elements produce far more accurate results for the same computational cost. In addition, higher modes lose their accuracy more rapidly in the nonhomogeneous case. One of the major causes is that there are too many spurious modes between the desired modes in nonhomogeneous case.

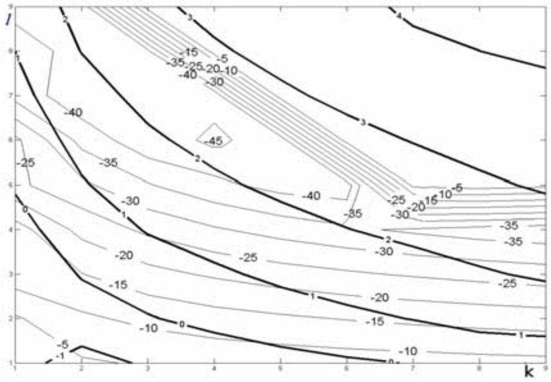

Fig. 4. Error (thin) and Computation Time (thick) contours for the 6^{th} TE^z mode eigen value vs k and l

TABLE I

CUT-OFF FREQUENCY OF SOME PRIME TE^z MODES IN THE HOLLOW WAVEGUIDE OBTAINED USING 64-NODE AND 4-NODE ELEMENTS(ELM) AND COMPARED TO ANALYTICALLY CALCULATED VALUES

Frequency$\times 10^{10} Hz$			TE_{ij}
Analytical	64-node ELM	4-node ELM	
0.74965845116532	0.74965845116536	0.75123230659503	01
1.49931690233064	1.49931690235604	1.51192959216360	02
2.24897535349596	2.24897581896715	2.29164773952789	03
0.49977230077688	0.49977230077672	0.50023846906262	10
0.90097732825383	0.90097732825389	0.90254556915563	11
1.24943075194220	1.24943075194483	1.25336134050884	21
2.12035429757121	2.12035429760716	2.13819133459099	32

TABLE II

CUT-OFF FREQUENCY OF SOME PRIME HYBRID MODES IN THE PARTIALLY LOADED WAVEGUIDE OBTAINED USING 64-NODE AND 4-NODE ELEMENTS(ELM) AND COMPARED TO ANALYTICALLY CALCULATED VALUES

Frequency$\times 10^{9} Hz$			Mode
Analytical	64-node ELM	4-node ELM	
4.55922365044621	4.559223650448166	4.569393605706706	1
6.06192092493685	6.061920924947108	6.073546202642527	2
10.4339453728603	10.43394577884192	10.57890756370606	3
10.9033473101955	10.90334786500416	11.04819184894751	4
15.5012752449882	15.50143821395703	15.70580243825406	5
19.5523926737524	19.85177079542511	2.043095257967376	6
19.8494789059400	25.48095312576569	2.557307218276380	7

VI. CONCLUSION

Arbitrary-degree mesh along with appropriate arbitrary degree fundamental element matrices has been developed and applied for analysis of two modal problems. Complexity and error analysis of the example problems reveal that arbitrary-degree high-order elements can afford considerable improvements in FEM analysis speed and accuracy. Thus, it is worth investing more algorithmic and coding efforts to more widely replace the regular lower order elements. Issues like arbitrary high-degree element mesh-generation and arbitrary high-order interpolation's of Tchebychev and Hermite[3] type are among the interesting ideas that need to be investigated. Ongoing work is looked into generation of similar results in 3D problems.

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