TWO DIMENSIONAL RECONSTRUCTION OF A DIELECTRIC CYLINDER BASED ON THE RECONSTRUCTION OF UNMEASURED EXTENDED T-MATRIX ELEMENTS

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1. Introduction

The inverse scattering problem of reconstructing a complex object in structure from the measured scattered waves is of interest from the viewpoint of application to medical diagnosis, underground prospection, and nondestructive examination. In numerous previous studies, nonlinear optimization techniques and regularizations are employed, where we suffer from traps at a local minimum and empirical decision on parameters. With the aim of decreasing the difficulty, some researchers dealt with the source type integral equation and investigated the nonradiating(unmeasured) equivalent current which contributes nothing to the scattered waves outside the object[1–3].

The authors have formulated the scattering problem using T-operator which transforms incident waves into the equivalent currents[4–6], and have proposed an iterative algorithm[7]. T-operator may be identified with the equivalent current for any incident waves. In this article, the algorithm is applied and extended to reconstructing an object in two dimensional structure also under noisy conditions. The scattered waves are reformed as elements of T-matrix, and their ineffective data due to noise are suppressed. The measured and unmeasured equivalent currents are separated by using orthonormal basis functions, where the measured equivalent currents are directly connected with the elements of T-matrix. The object and the unmeasured equivalent current are reconstructed by decreasing a residual error of the equivalent current in the least square approximation without solving the direct problem and an additional regularization.

2. Formulation

Let us consider a scattering problem of a cylindrical object located in a region R_{V} of free space under E-wave time-harmonic excitations. The geometry is shown in Fig. 1. The time factor $\exp(\mathrm{j}\omega t)$ is suppressed hereafter. The object is described by the object function $\chi(\boldsymbol{r}'') = \varepsilon_{\mathrm{r}}(\boldsymbol{r}'') - 1$, where ε_{r} is the dielectric constant. We denote the scattered wave by u_{s}, the incident wave by u_{in}, and the total wave by u_{t}. These waves satisfy the integral equations:

$$u_{\mathrm{s}}(\boldsymbol{r}) = \int_{R_{\mathrm{V}}} G(\boldsymbol{r}, \boldsymbol{r}'') J_{\mathrm{eq}}(\boldsymbol{r}'') \mathrm{d}\boldsymbol{r}'', \qquad \boldsymbol{r} \notin R_{\mathrm{V}} \tag{1}$$

$$u_{\mathrm{t}}(\boldsymbol{r}') = u_{\mathrm{in}}(\boldsymbol{r}') + \int_{R_{\mathrm{V}}} G(\boldsymbol{r}', \boldsymbol{r}'') J_{\mathrm{eq}}(\boldsymbol{r}'') \mathrm{d}\boldsymbol{r}'', \qquad \boldsymbol{r}' \in R_{\mathrm{V}} \tag{2}$$

where G is Green's function given by $G(\boldsymbol{r}', \boldsymbol{r}'') = -\frac{\mathrm{j}}{4} \mathrm{H}_0^{(2)}(k|\boldsymbol{r}' - \boldsymbol{r}''|)$, k is the wavenumber in free space, $\mathrm{H}_n^{(2)}$ is the Hankel function of the second kind of order n, and $J_{\mathrm{eq}}(\boldsymbol{r}')$ is the equivalent current defined by

$$J_{\mathrm{eq}}(\boldsymbol{r}'') = k^2 \chi(\boldsymbol{r}'') u_{\mathrm{t}}(\boldsymbol{r}''). \tag{3}$$

ISBN: 89-86522-79-9 94460©KEES

The incident wave is a plane wave propagating in θ direction and is written by

$$u_{\rm in}(\boldsymbol{r};\theta) = \sum_{n=-\infty}^{\infty} \alpha_n(\theta)\mathrm{J}_n(k\rho)\exp(\mathrm{j}n\phi), \qquad \alpha_n(\theta) = \mathrm{j}^{-n}\exp(-\mathrm{j}n\theta) \tag{4}$$

where J_n is the Bessel function of order n. The scattered wave in far region is written by

$$u_{\rm s}(\boldsymbol{r}_{\rm s};\theta) \sim \sqrt{\frac{2\mathrm{j}}{\pi k\rho_{\rm s}}}\exp(-\mathrm{j}k\rho_{\rm s})\bar{u}_{\rm s}(\phi_{\rm s};\theta), \qquad \bar{u}_{\rm s}(\phi_{\rm s};\theta) = \sum_{m=-\infty}^{\infty} \beta_m(\theta)\,\mathrm{j}^m\exp(\mathrm{j}m\phi_{\rm s}). \tag{5}$$

The coefficients $\beta_m(\theta)$ and $\alpha_n(\theta)$ is related by so-called T-matrix[8] as $\beta_m(\theta) = \sum_{n=-\infty}^{\infty}\tau_{mn}\alpha_n(\theta)$. Using the second equation of (4), we get

$$\beta_m(\theta) = \sum_{n=-\infty}^{\infty} \tau_{mn}\,\mathrm{j}^{-n}\exp(-\mathrm{j}n\theta)\,. \tag{6}$$

From (5) and (6), we know that the measured scattered waves are transformed into the T-matrix elements.

Let $R_{\rm V}$ be the circular region of $|\boldsymbol{r}''| \le b$. We define the inner product on $R_{\rm V}$ by $\langle f(\boldsymbol{r}), g(\boldsymbol{r})\rangle = \int_{R_{\rm V}} f^*(\boldsymbol{r})g(\boldsymbol{r})\mathrm{d}\boldsymbol{r}$, where the asterisk denotes the complex conjugate, and introduce the orthonormal set of functions with two indices:

$$\eta_{m,m'}(\boldsymbol{r}'') = \frac{1}{\sqrt{2\pi c_{m,m'}}}\mathrm{J}_m(k_{m,m'}\rho'')\exp(\mathrm{j}m\phi''), \qquad \begin{array}{l} m = 0, \pm1, \pm2, \cdots \\ m' = 1, 2, \cdots, \mathrm{I}(m), \cdots \end{array} \tag{7}$$

Here, $c_{m,m'}$ is defined for normalization as $c_{m,m'} = \int_0^b \mathrm{J}_m^2(k_{m,m'}\rho'')\rho''\mathrm{d}\rho''$, and $k_{m,m'}$ are determined so that $\{\eta_{m,m'}\}$ forms an orthogonal set, $k_{m,m'} < k_{m,m'+1}$, and only one coefficient $k_{m,I(m)}$ is equal to k.

Let us suppose that incident waves are given by $u_{\rm in}^{(n)}(\boldsymbol{r}) = \eta_{n,\mathrm{I}(n)}(\boldsymbol{r});\quad (n = 0, \pm1, \pm2, \cdots)$, and express the equivalent current approximately by

$$J_{\rm eq}^{(n)}(\boldsymbol{r}') \approx \sum_{m=-M}^{M}\sum_{m'=1}^{M'} \eta_{m,m'}(\boldsymbol{r}')\; w(m,m',n) \tag{8}$$

where $w(m,m',n)$ is the expansion coefficient. Comparing (1) with (5) using (8) and (6), we get

$$w(m,\mathrm{I}(m),n) = \frac{2\mathrm{j}}{\pi\sqrt{c_{m,\mathrm{I}(m)}c_{n,\mathrm{I}(n)}}}\tau_{mn}. \tag{9}$$

From the above, we can see the expansion coefficient $w(m,m',n)$ as an extension of the T-matrix, and know that a part of the coefficient $w(m,\mathrm{I}(m),n)$ is measured one which is related to the scattered wave and the others $w(m,m',n);\ m' \ne \mathrm{I}(m)$ are unmeasured ones which cannot be observed from the scattered waves.

We introduce another set of orthogonal functions $\{\psi_l\}$ $(l = 1,\cdots,L)$ over $R_{\rm V}$ to expand the object function as

$$\chi(\boldsymbol{r}) = \sum_{l=1}^{L} e_l\,\psi_l(\boldsymbol{r}). \tag{10}$$

The cost functional of the object function and the unmeasured equivalent current is defined by

$$\Omega(\boldsymbol{e},\boldsymbol{w}_{\rm u}) = \sum_{n=-N}^{N}\left\|J_{\rm eq}^{(n)}(\boldsymbol{r}) - k^2\chi(\boldsymbol{r})u_{\rm t}^{(n)}(\boldsymbol{r})\right\|^2 \to \min \tag{11}$$

where $\boldsymbol{e}$ and $\boldsymbol{w}_{\rm u}$ indicate $\{e_l\}$ and $\{w(m,m',n);\ m' \ne \mathrm{I}(m)\}$, respectively, $u_{\rm t}^{(n)}(\boldsymbol{r}')$ is the total wave generated by $J_{\rm eq}^{(n)}(\boldsymbol{r}'')$ in $R_{\rm V}$, and $\|f(\boldsymbol{r})\|^2 = \langle f(\boldsymbol{r}), f(\boldsymbol{r})\rangle$. We can reduce the inverse scattering problem to minimization of (11) to find the optimal coefficients $\boldsymbol{e}$ and $\boldsymbol{w}_{\rm u}$.

At first, assuming that the object $\boldsymbol{e}$ is given, we get a linear equation for unmeasured coefficients of the equivalent current $\boldsymbol{w}_{\rm u} = \{w(m,m',n);\ m' \ne \mathrm{I}(m)\}$ as

$$\sum_{m=-M}^{M} \sum_{\substack{m'=1 \\ m' \neq \mathrm{I}(m)}}^{M'} A(p,p',m,m')w(m,m',n) = B(p,p',n), \quad (p=-M,\cdots,M;\ p'=1,\cdots,M') \quad (12)$$

where $A(p,p',m,m')$, $B(p,p',n)$ are determined from the given object $\boldsymbol{e}$ and the measured coefficients of the equivalent current.

Next, assuming that the equivalent current is given, we get a linear equation for the expansion coefficients of the object $\boldsymbol{e} = \{e_l\}$ as

$$w(p,p',n) = \sum_{l=1}^{L} e_l C(l;p,p',n), \qquad (p=-M,\cdots,M;\ p'=1,\cdots,M';\ n=-N,\cdots,N) \quad (13)$$

where $C(l;p,p',n)$ is determined from the given equivalent current.

Equations (12) and (13) are solved as the linear least-squares problem using the QR decomposition. The inverse algorithm is summarized as follows:

Step 1: Set the initial value of $\boldsymbol{e}$.
Step 2: Update $\boldsymbol{w}_{\mathrm{u}}$ by (12).
Step 3: Update $\boldsymbol{e}$ by (13), and go back to Step 2 and repeat Steps 2 and 3 until convergence.

3. Numerical Examples

We can know from Parseval's formula that error in $|\tau_{mn}|^2$ becomes approximately $P_{\mathrm{N}}/(N_{\mathrm{i}}N_{\mathrm{s}})$ when averaged noise-power relative to the power of incident plane wave is P_{N}, where N_{i} and N_{s} are the number of plane wave incidences and that of observation directions of the scattered wave, respectively. Figure 2 shows the exact T-matrix elements and the reconstructed ones for a dielectric circular cylinder of radius 0.8λ and $\varepsilon_{\mathrm{r}} = 1.8$, where $\lambda = 2\pi/k$ is a wavelength in free space, and $P_{\mathrm{N}} = 10^{-2}$, $N_{\mathrm{i}} = N_{\mathrm{s}} = 100$ are assumed. The T-matrix elements for $|m| \leq 7$ and $|n| \leq 7$, which are larger than 10^{-6}, are effective against noise. In this case we have assumed $M = N = 7$. The number of independent T-matrix elements becomes 120 in consideration of the symmetry of T-matrix[9].

The region R_{V} is divided into small cells by $D \times D$(Fig. 1). Let $\psi_l(\boldsymbol{r}'')$ be the pulse function such that $\psi_l(\boldsymbol{r}'') = 1$ in the l-th cell; otherwise $\psi_l(\boldsymbol{r}'') = 0$, where the cells out of R_{V} are excluded. The number of cells L becomes 112 for $D = 12$, and $L = 156$ for $D = 14$. Figure 3 shows the cost functional Ω and the residual errors of the reconstructed object E_χ as functions of the iteration number; and Fig. 4 shows the reconstructed profiles after 50 iterations. As seen in Fig. 4(c), if L is smaller than the number of independent T-matrix elements, fatal profile error due to noise does not appeared. In Fig. 4(d), we can see that profile error increases because L is larger. From Fig. 4(e), we know that the number of terms of the equivalent current $(2M+1) \times M'$ should be no less than L. The computation time for 50 iterations for Fig. 4(c) was 50 minutes by Compaq W8000 with Xeon 2.4GHz processor.

4. Concluding Remarks

In this article, we have introduced a T-matrix expression of the scattered wave and expressed the equivalent current in terms of orthonormal basis functions. Using the expressions, we have formulated the inverse scattering problem of reconstructing a two dimensional object. As a result, we can directly connect the noise-removed scattered waves to the measured equivalent current. We have proposed an iterative algorithm that the object and the unmeasured equivalent current are updated by decreasing the cost functional in the least square approximation. The algorithm avoids employing a nonlinear optimization algorithm, solving the direct scattering problem, and using a special additional regularization. Numerical examples show that the algorithm works well under noisy conditions. An extension of it to the use of multi-frequencies is a future subject.

Acknowledgement

This work was supported in part by Grant-in-Aid for Young Scientists (B), 14750351, from the Ministry of Education, Culture, Sports, Science and Technology.

References

[1] T. M. Habashy, M. L. Oristaglio, and A. T. de Hoop, **29**(4), 1101–1118, 1994.
[2] P. M. van den Berg and R. E. Kleinman, **13**, 1607–1620, 1997.
[3] S. Caorsi and G. L. Gragnani, *Radio Sci.*, **34**(1), 1–8, 1999.
[4] K. Ishida and M. Tateiba, *Progress in Electromagnetics Research Sympo. Proc.*, 25, 1998.
[5] K. Ishida, H. Furukawa, and M. Tateiba, *Proc. 2000 ISAP*, **3**, 1299–1302, 2000.
[6] K. Ishida and M. Tateiba, *Proc. 2004 URSI Int. Sympo. on Electromagnetic Theory*, 912–914, 2004.
[7] K. Ishida and M. Tateiba, *Proc. 2004 ISAP* 593–596, 2004.
[8] P. C. Waterman, *J. Acoust. Soc. Am.*, **45**(6), 1417–1429, 1969.
[9] O. M. Bucci and T. Isernia, *Radio Sci.*, **32**(6), 2123–2137, 1997.

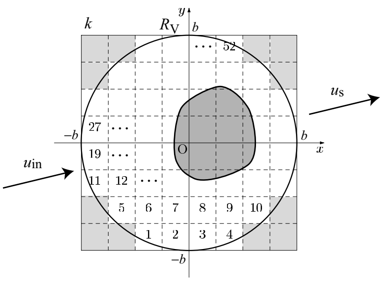
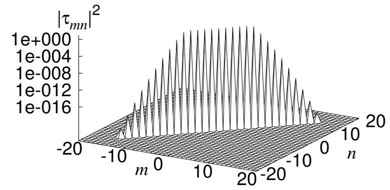
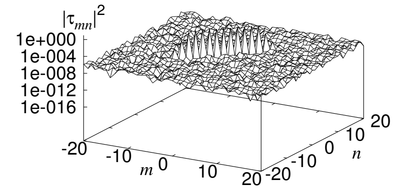


Figure 1: Geometry of the problem.



(a) exact



(b) under $P_{\mathrm{N}} = 10^{-2}$ noise

Figure 2: T-matrix elements reconstructed from scattered waves. The radius of the object is 0.8λ, ε_{r}=1.8.

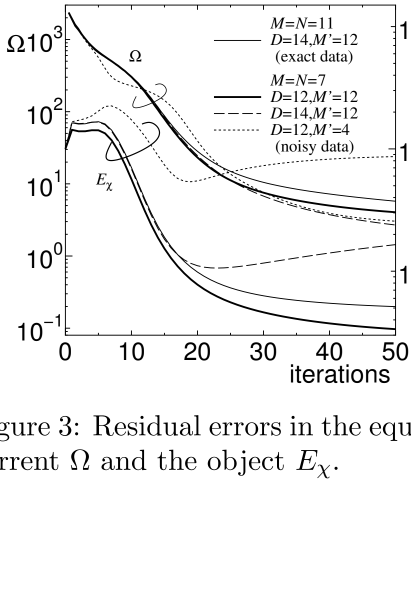
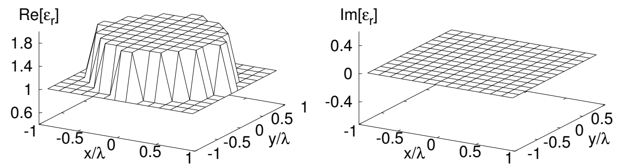
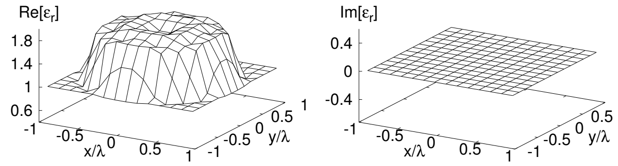


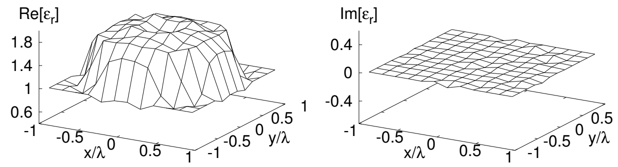
Figure 3: Residual errors in the equivalent current Ω and the object E_χ.



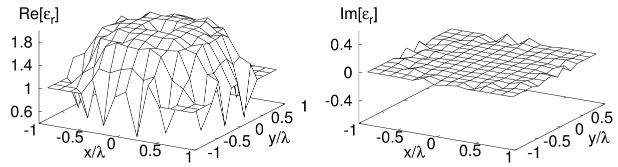
(a) original



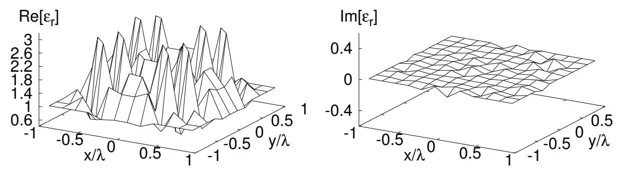
(b) reconstructed from the exact scattered wave, $M = N = 11,\ D = 14,\ M' = 12$



(c) reconstructed from scattered wave with 10^{-2} noise relative to the incident wave power, $M = N = 7,\ D = 12,\ M' = 12$



(d) as (c), but $D = 14,\ M' = 12$



(e) as (c), but $D = 12,\ M' = 4$

Figure 4: Reconstructed profiles after 50 iterations. The original object is of 0.8λ radius and ε_{r}=1.8.