AN APPROACH TO A NUMERICALLY WELL-POSED SOLUTION TO THE INVERSE SCATTERING PROBLEM

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Traditionally, inverse scattering problems treated using an exact formulation have been viewed as being so ill-posed that their solutions are rarely practical in "real-world" applications. Several years ago, in work that has not been published, the author derived some rather simple relations which pinpointed where the ill-posedness occurs in this solution, and which indicated that the physical requirement that the source be of finite size ("compact support") should be a sufficient constraint to remove the ill-posedness. These relations did not result in an evident method for implementing a stable numerical solution. This paper begins by reviewing these results. It is then shown that the solution to the inverse scattering problem as formulated is mathematically the same as a common problem in digital image restoration. Indeed, there are also some interesting physical parallels between the two problems. Using results from this field, it is shown that a solution in the minimum mean square error sense exists, and that this solution is both inherently well-posed and numerically efficient and stable. A second formulation, which permits any additional constraints which are known a priori to be readily incorporated, is also presented. Because of space limitations, many important aspects of the results cannot be treated in detail. Interested workers are urged to contact the author for additional information.

In an inverse scattering problem, the fields in the inhomogeneous wave equation are assumed known by measurement, and it is desired to solve for the source term. The source term may represent an actual source distribution, or it may include equivalent sources produced by illuminating a scattering object with an incident, or probing, field. In the latter case, the incident field is assumed known. Although this paper uses scalar notation, all of the results presented have been shown to apply to the general, full vector field and tensor medium quantities. Consider a source $\rho(\underline{x},\omega)$ in a domain D bounded by a surface S. Then the time-harmonic field, $\phi(\underline{x},\omega)$, due to $\rho(\underline{x},\omega)$ is the solution to the inhomogeneous wave equation

$$\nabla^2\Phi + K^2\Phi = -\rho \qquad (1)$$

where $k = 2\pi/\lambda$. $\phi(x,\omega)$ is measured over some surface enclosing the source, and the purpose is to determine $\rho(x,\omega)$. If $n(x,\omega)$ is the complex refractive index of a scattering region, then the equivalent source is

$$\rho(\underline{x}) = K^2\left[N^2(\underline{x}) - 1\right]\Phi(\underline{x}) \qquad (2)$$

The basic exact integral equation approach to this problem considered in this paper was derived in reference [1], and, in a different form, in reference [2] (see also reference [3]). Let a field quantity ϕ_H be defined by

$$\Phi_H(\underline{x}) = \oint dS\,(G^*\nabla\Phi - \Phi\nabla G^*) \qquad (3)$$

where G^* is the complex conjugate of the free space Green's function. This equation is the mathematical statement of the field that is reconstructed from a hologram recorded on the surface S. Because it depends only on the field which can be measured over the surface S, it is known. Using this field, the following integral equation can be derived [1, 2, 3]:

$$\Phi_H(\underline{x}) = 2i\int dV'\ \mathrm{Im}\,G(\underline{x}-\underline{x}')\ \rho(\underline{x}') \quad + \ \Phi_I(\underline{x}) \qquad (4)$$

where Im G is the imaginary part of the Green's function. This integral equation must be solved for ρ. This is a Fredholm integral equation of the first kind, which is often identified as inherently ill-posed.

The nature of the ill-posedness, and why the compact support of ρ should permit this to be overcome can be shown as follows. Without loss of generality, the incident field term will be dropped from equation (4) in what follows. Let a superscript tilde denote the spatial Fourier transform of a function. Then, noting that the integral in equation (4) is convolution and denoting the imaginary part of G as G_i, the spatial Fourier transform of equation (4) becomes

$$\tilde{\Phi}_H = 2i\ \tilde{G}_I\tilde{\rho} \qquad (5)$$

and the obvious solution is

$$\tilde{\rho} = (2i)^{-1}\tilde{\Phi}_H/\tilde{G}_I \qquad (6)$$

Unfortunately, G_i contains many zeros. Unless ϕ_H goes to zero at the same points at least as rapidly as does G_i, this solution contains singularities.

It can be shown that, theoretically, $\tilde{\rho}$ should not contain singularities. A completely rigorous proof of this requires more space than is available, because of the care which must be paid to the elements of the theory of distributions involved. However, the proof is easily outlined. Let ρ be of compact support, and let it be assumed that $\tilde{\rho}$ is singular. It follows that $\tilde{\rho}$ can be written as the sum of a nonsingular part and a part containing all of the singularities. Furthermore, these singularities can be written as a summation over the points in the (transform) domain at which they occur, with the singularity at each point written as the product of some weighting function times a delta distribution (and/or derivatives of delta distributions). Because of the linearity of the wave equation, it follows that the part of ρ containing all of the singularities is the inverse spatial Fourier transform of this summation. But ρ, and, in particular, the singular part of ρ, must be of compact support. However, the inverse spatial Fourier transform of the summation over the delta distributions cannot be of compact support, by the nature of the transform of such a distribution. Thus, the assumption that $\tilde{\rho}$ contains a singular part must be false. It follows that wherever the denominator of equation (6) goes to zero, the numerator must go to zero at least as rapidly. When this does not happen, it must be due to errors (noise, numerical errors, sampling effects, etc.) in the numerator.

It can also be proven that the solution given by equation (6) should be well-posed. Let $\rho = \rho + \rho_N$, where the second term contains all sources of error. Let a signal-to-noise ratio be defined as ϵ, where $|\tilde{\rho}_N| \leq \epsilon |\tilde{\rho}|$. Then, the solution will be well-posed if the upper bound on ρ_N is $\leqslant \epsilon$ times the upper bound on ρ, for all points in the solution domain. By a basic theorem of Fourier transforms [4],

$$|\rho_N| \leq \int |\tilde{\rho}_N| \, D\underline{r} \tag{7}$$

But $|\tilde{\rho}_N| \leq \epsilon |\tilde{\rho}|$, and thus

$$|\rho_N| \leq \int \epsilon |\tilde{\rho}| = \epsilon \int |\tilde{\rho}| \, D\underline{r} \tag{8}$$

Again, using the above Fourier transform theorm, it follows that

$$|\rho| \leq \int |\tilde{\rho}| \, D\underline{r} \tag{9}$$

Thus, the upper bound on the noise contribution to the solution is $\leqslant \epsilon$ times the upper bound on the solution, and the solution should be stable. Using similar proofs, it can be shown that the spatial derivatives of the source term should be similarly stable. The definition of signal-to-noise ratio used here is actually a more stringent one than is required. What it says is that there should be an upper bound on the noise contribution to each spatial frequency component of the source term. The proof still holds if the normal, spatial-domain signal-to-noise ratio is employed, and, using the "energy" theorem for Fourier transforms, an upper bound for each spatial frequency component is derived. Furthermore, as is true of all of the relations derived in this paper, a temporal Fourier transform may be taken over all temporal frequency components to obtain analgous time domain results.

Unfortunately, while the above discussion indicates that it should be possible to use equation (6) to obtain a well-posed solution to the inverse scattering problem, it does not tell how to do this. The theory of digital image restoration provides one method. The mathematical representation of an image degraded by a linear process (e.g., motional blur) is identical to the form of equation (4): the holographic field represents the degraded image, the Green's function is the degrading function, and the source term is the undegraded image. [Note that there is even a valid analogy to be drawn between the physics of the two problems: the process of propagation of a field can be viewed as a "degrading" of the original field. A clear understanding of this analogy requires a discussion of the fact that the holographic field is proportional to the source term itself in the short wavelength limit [5], which is beyond the scope of this paper.] The technique of minimum mean square error restoration [6] will be used. Since the operations of equation (4) are linear, a linear estimate of the solution is chosen. The problem is then to minimize the error, in the mean square sense, between the actual value of ρ plus a noise term, and the value of obtained by applying a linear operator to the holographic field. The algebra of doing this requires several pages, but the result [6] permits the linear operator to be expressed in terms of the known data and an estimate of the noise. The result is a rather complicated matrix expression which would be quite costly to evaluate. However, the imaginary part of the Green's function in equation (4) is a Toeplitz (or block Toeplitz) matrix, when discretized. By approximating this matrix as a circulant matrix, the complete result can be reduced to a form in which only algebraic and fast Fourier transform (FFT) operations are necessary for evaluation. The result, for two dimensions, is given by

$$\tilde{\rho}(\nu_X, \nu_Y) = \left[\tilde{G}^*(\nu_X, \nu_Y)\, \tilde{\Phi}_H(\nu_X, \nu_Y)\right] \Big/ \left\{ |\tilde{G}(\nu_X, \nu_Y)|^2 + \left[\tilde{P}_N(\nu_X, \nu_Y)/\tilde{P}(\nu_X, \nu_Y)\right] \right\} \tag{10}$$

where P_n and P are the power spectra of the noise (errors) and of the exact solution. Several important points must be made about this solution. It is always well conditioned, since although the first term in the denominator is known to have zeros, the second term can never fall below the ratio of the power spectras. If the noise goes to zero, equation (10) reduces to equation (6), the solution originally obtained. As the source power spectrum approaches zero, so does the solution. This is reasonable, since recovery of information at spatial frequencies where the source information is dominated by noise is not reasonable. Finally, it should be pointed out that a related solution can be derived by minimizing a slightly different quanity, namely, the quanity minimized in deriving equation (10) but with the addition of a scalar constraint parameter, (see reference [6] for a derivation):

$$\tilde{\rho}(\nu_X, \nu_Y) = \left[\tilde{G}^*(\nu_X, \nu_Y)\, \tilde{\Phi}_H(\nu_X, \nu_Y)\right] \Big/ \left\{ |\tilde{G}(\nu_X, \nu_Y)|^2 + \gamma |\tilde{P}_N(\nu_X, \nu_Y)|^2 \right\} \tag{11}$$

As might be expected from the above comments, it can be shown that an optimum value of γ can be derived from an estimate of the mean and variance of the error (noise) present in the data. Note that an upper bound to this quanity is usually known in a real-world measurement.

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