Infinite Current Behavior along a Subwavelength Perfectly Conducting Concaved Wedge

Dr Ir Thierry Gilles
Department CISS, Royal Military Academy, BRUSSELS, Belgium

Abstract – **Equivalent surface currents are known to grow to infinity close to a sharp edge. In the Method of Moments, the linear or higher order basis functions representing those currents are always finite. Special basis functions have been derived, that allow infinite currents on sharp edges, but their efficient use requires to know the exact analytical behavior. In this paper we derive this exact behavior for a perfectly conducting subwavelength concaved wedge.**

Index Terms — **Concaved, wedge, subwavelength, current**

1. Introduction

Sommerfeld solved the half plane illuminated by a plane wave [1], and Mac Donald generalized this result to the PEC wedge [2]. Those exact analytical results show that the component of the total fields parallel to an edge are always finite, but that the component of the total fields perpendicular to an edge may become infinite. Exact solutions have also been obtained for an infinity [3][4] or for only two [5][6][7] parallel stacked half planes. The general conclusion of all these works is that the singular fields grow like $\rho^{-\nu}$ very close to the edge, where ρ is the distance to the edge, and where ν depends both on the geometry and the electromagnetic properties of the wedge-shaped sectors surrounding the edge. Much less effort is made to obtain the exact behavior of these current densities in the vicinity of the more or less coupled edges.

The purpose of this paper is to derive the explicit law for the growth to infinity of the current density along the concaved edge of a subwavelength concaved wedge. Knowledge of this behavior can for example be incorporated in specialized sharp edge basis functions used in the Method of Moments.

2. General concaved wedge theory

In [8] a solution is given for the concaved wedge depicted on Fig.1 illuminated by a line source. It is based on a radial mode matching technique leading to an infinite set of linear equations that can fortunately be truncated after a few terms only. In this paper we modify this analysis to the TM plane wave excitation case $\bar{E}_i = \hat{z}e^{jk_0(x\cos\phi_i + y\sin\phi_i)}$, that can be easily derived from the line source case by letting it recede to infinity. The time convention is $e^{+j\omega t}$ thus the Hankel functions are all of the second kind.

Starting from the general description of the solution method given in [8], we will first obtain the J_z (= H_ϕ) current density all along the concaved edge.

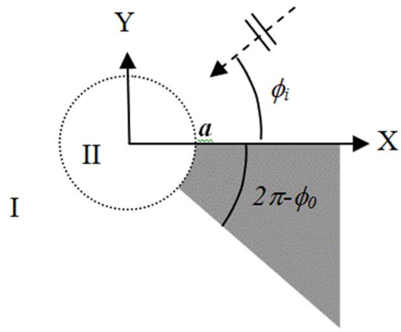

Fig. 1. Geometry for the concaved edge.

If we do not use the Wronskian, as proposed in [8], the infinite square system to solve becomes :

$$\sum_{p=1}^{\infty} \underbrace{\left\{ \delta_{pq} - \frac{H_\mu(ka)}{H'_\mu(ka)} I_{qp}(ka) \right\}}_{HH} B_p H'_\mu(ka) = -\sum_{p=1}^{\infty} \underbrace{\left\{ \delta_{pq} - \frac{J_\mu(ka)}{J'_\mu(ka)} I_{qp}(ka) \right\}}_{JJ} s_p j^\mu J'_\mu(ka) \quad (1)$$

In matrix and vector notation (p,q = 1,2,..∞) :

$$\left(B_p H'_\mu(ka) \right) = \underbrace{-[HH]^{-1}[JJ]}_{[HJ]} \left(s_p j^\mu J'_\mu(ka) \right) \quad (2)$$

where :

$\mu = p\pi/\phi_0$ $\qquad \nu = q\pi/\phi_0$

$s_p = 4\pi \sin(\mu\phi_i)/\phi_0$

$G_{n\mu} = \int_0^{\phi_0} \sin(\mu\phi) e^{-jn\phi}.d\phi$ $\qquad G_{n\nu} = \int_0^{\phi_0} \sin(\nu\phi) e^{jn\phi}.d\phi$

$I_{qp} = \frac{1}{\pi\phi_0} \sum_{n=-\infty}^{\infty} \frac{J'_n(ka)}{J_n(ka)} G_{n\mu} G_{n\nu}$ $\qquad \delta_{qp}$ *is the Kronecker delta*

Once the B_p unknowns are determined by solving (2), the A_n unknowns [8] follow (n = 1,2,..∞) by :

$$A_n = \frac{1}{2\pi J_n(ka)} \sum_{p=1}^{\infty} \left\{ s_p j^\mu J_\mu(ka) + B_p H_\mu(ka) \right\} . G_{n\mu} \quad (3)$$

3. Subwavelength concaved wedge

For small ka (< 0,01), the matrices [HH], $[HH]^{-1}$, [JJ], and consequently [HJ] are all real, have a chessboard structure (the elements with indices p,q such that p+q is odd are identically zero), depend only on the wedge angle ϕ_0 and become independant of ka. Indeed :

$$\frac{J_\mu(ka)}{J'_\mu(ka)} I_{qp} = -\frac{H_\mu(ka)}{H'_\mu(ka)} I_{qp} \overset{ka<<\sqrt{\mu}}{\cong} \frac{4\nu}{\pi\phi_0} \sum_{n=1}^{\infty} n. \frac{1-\cos\left[(n+\mu)\phi_0\right]}{\left(n^2-\mu^2\right)\left(n^2-\nu^2\right)} \tag{4}$$

Also every elements of $[HH]^{-1}$, [JJ] and [HJ] are comprised between 0 and 1.

All these properties, valid for small ka, allow to demonstrate that the B_p unknowns are decreasing extremely rapidly, as $\{ka\}^{\mu+\mu_1}$ for odd p and $\{ka\}^{\mu+\mu_2}$ for even p. These two different laws for odd and even p stem from the chessboard structure of the matrix [HJ]. A numerical example is given in Table 1.

TABLE I

B_p for ka = 0,01 / ϕ_0 = 70° / ϕ_i = 30°

B_1	(836,4 – j 572,2) .1e-06
B_2	(389,8 + j 155,4) .1e-09
B_3	(423,7 + j 289,1) .1e-09
B_4	(252,1 + j 100,6) .1e-12
B_5	(115,9 + j 79,0) .1e-12

The clearly dominant term B_1 can be written as :

$$B_1 = K_1(\phi_0).\sin[\pi\phi_i/\phi_0](ka)^{2\pi/\phi_0} \tag{5}$$

where K_1 is a function depending only of ϕ_0.

The total magnetic field in zone II, especially for ρ = a, can then be determined by [8] :

$$H_\phi^{II}(\rho=a,\phi) = j.YE_0 \sum_{n=-\infty}^{\infty} A_n \left.\frac{\partial J_n(k\rho)}{\partial(k\rho)}\right|_{\rho=a} e^{jn\phi} \tag{6}$$

For small ka, using (3) and (5), (6) reduces to :

$$H_\phi^{II}(\rho=a,\phi) \overset{ka<0,01}{\cong} YE_0 K_1(\phi_0)(2.ka)^{\pi/\phi_0-1}.P(\phi,\phi_0) \tag{7}$$

where :

$$P(\phi,\phi_0) = \sum_{n=1}^{\infty}\left\{2n.\left(\frac{\cos[n(\phi-\phi_0)]+\cos\left[n(2\pi-\phi)\right]}{n^2-(\pi/\phi_0)^2}\right)\right\} \tag{8}$$

This final expression allows to derive the two main properties of the current density along the concaved edge. Firstly its amplitude all along the edge varies as $\{ka\}^{\pi/\phi_0-1}$. For $\phi_0 \le \pi$, a reentrant wedge or a flat opened plane, the current density in the concaved edge reduces to zero if ka tends to zero, while for $\phi_0 > \pi$ it increases to infinity. For a 90° concaved wedge (ϕ_0 = 3π/2), this law is $\{ka\}^{-1/3}$ and the limit law for a half plane (ϕ_0 = 2π), where the concaved edge actually disappears, would be $\{ka\}^{-1/2}$.

The current density distribution all along the concaved edge, described by (8), is independent of ϕ_i and has a U shape, as seen in Fig.2. The distribution is minimum in the middle of the edge and infinite at the extremities.

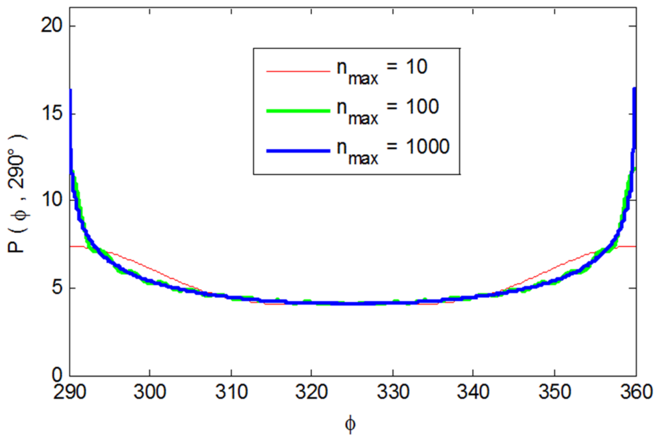

Fig. 2. U shape law for the concaved edge.

4. Conclusion

The exact analytical behavior of the current density inside the concaved edge of a subwavelength perfectly concaved wedge has been established. As expected the current density is infinite on the edges, while the U shaped distribution of the current amplitude along the concaved edge is shown to be symmetrical and independent of the plane wave incident angle.

References

[1] A. Sommerfeld, Math. Anal., No. 47, pp. 317, 1896.

[2] H.M. Macdonald, Electric Waves, *The University Press*, Cambridge, England, pp.186-198, 1902.

[3] A.E.Heins and J.F.Carlson, "The Reflection of an Electromagnetic Plane Wave by an Infinite Set of Plates, I.", *Quart. Appl. Math.* No 4, pp. 313-329, 1947.

[4] A.E.Heins and J.F.Carlson, "The Reflection of an Electromagnetic Plane Wave by an Infinite Set of Plates, II.", *Quart. Appl. Math.* No 5, pp. 82 88, 1947.

[5] A.E.Heins, "The Radiation and Transmission Properties of a Pair of Semi Infinite Parallel Plates I", *Quart. Appl. Math.* No 6, pp. 157 166, 1948.

[6] A.E.Heins, "The Radiation and Transmission Properties of a Pair of Semi Infinite Parallel Plates II", *Quart. Appl. Math.* No 6, pp. 215-220, 1948.

[7] L.A.Vajnshtejn, "On the Theory of Diffraction by Two Parallel Half-Planes,", *Izv. Akad. Nauk.*, Ser. Fiz. No 12, pp. 166 180, 1948.

[8] J-W.Yu and N-M Myung, "TM scattering by a wedge with concaved edge", *IEEE Transactions on Antennas and Propagation*, vol. 45, No 8, pp. 1315-1316, August 1997.

[9] J.L.Volakis, A.Chatterjee and L.Kempel, *Finite Element Method for Electromagnetics*, Piscataway, NJ : IEEE Press, 1998..