FIELD SINGULARITIES AT WEDGES AND TIPS OF CONES

J. Van Bladel and R. Desmedt
Laboratory of Electromagnetism and Acoustics
University of Ghent, Ghent, B9000 Belgium

A knowledge of singularities is important for the evaluation of the fields, as the singularity can be incorporated in the numerical algorithm to increase the latter's speed of convergence. Further, it allows one to check the validity of a theoretical solution by verifying the field behavior in the vicinity of the singularity. Finally, the singular behavior must be known to evaluate possible dangers of breakdown.

It is sufficient, in a study of the singular field behavior, to only consider distances to the edge which are small with respect to the wavelengths of interest. For such case the problem becomes a static one. In the case of the dielectric wedge of Fig. 1 the theory is well-documented [1,2]. Two basic singularities must be considered. The first is associated with symmetric potentials

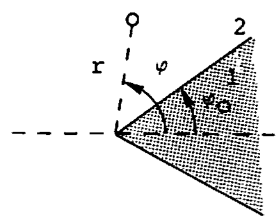

Fig. 1

$$\phi_1 = Ar^{\nu} \cos \nu\varphi \qquad (1)$$

$$\phi_2 = A \frac{\cos \nu\varphi_o}{\cos \nu(\pi-\varphi_o)} r^{\nu} \cos \nu(\pi-\varphi)$$

For such symmetry, the $\varphi = 0$ plane represents a magnetic wall. The boundary conditions at $\varphi = \varphi_o$ quantize ν. Only values of ν below unity create singularities in the electric field $\bar{e}$, as the latter's components are proportional with $r^{\nu-1}$. The relevant values of ν are shown in Fig. 2; they occur only for a sharp wedge.

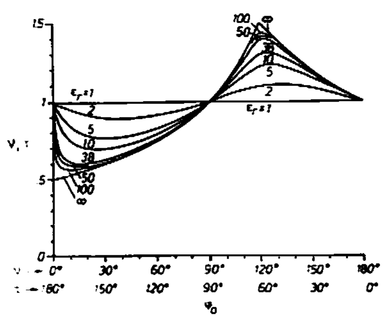

Fig. 2

The second symmetry concerns anti-symmetric potentials, for which the $\varphi = 0$ plane behaves as an electric wall. Field singularities now occur for reentrant wedges. The potential is proportional with r^{τ}, where τ is shown in Fig. 2. An arbitrary potential will excite the ν or τ singularities through its symmetric or anti-symmetric components.

The considerations given above also hold for the magnetic field, provided ε_r is replaced by μ_r. In the limit $\mu_r \to \infty$, i.e. for perfect iron, the exponent ν for a sharp wedge is $\pi/2(\pi-\varphi_o)$, and for a reentrant wedge it is $\pi/2\varphi_o$. The lines of force of $\bar{b}$ are perpendicular to the iron on the air side, but make a well-defined angle, on the iron side, with the boundaries $\varphi = \pm \varphi_o$ [3].

Figure 3 shows structures which incorporate both dielectric

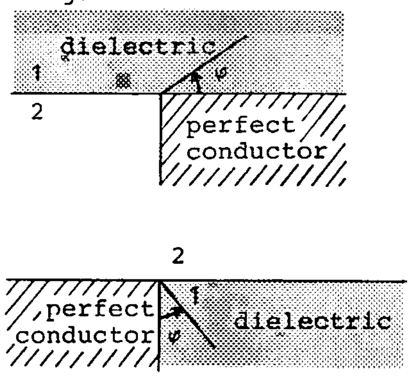

Fig. 3

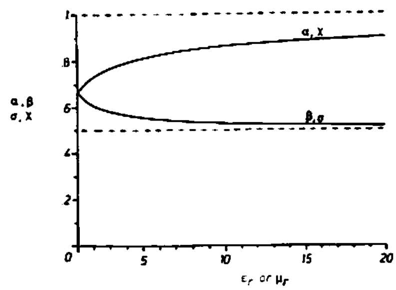

Fig. 4

and metallic wedges. These types of wedge are found in dielectric resonators, waveguides, microstrips, quartz windows etc... The corresponding singularity exponents for the electric field (α for Fig. 3a, α for Fig. 3b) are shown in Fig. 4, together with the corresponding values χ, β for the magnetic field, which is tangent to the metal. Extensive additional details on the structures of Fig. 1 and 3 are given in [4]. They concern equipotentials, lines of force and formulas for the fields. The theory is based on potentials of the general form (1).

Metallic circular cones are found in e.g. sharp needles, lightning arresters and rocket models (Fig. 5). Two kinds of singularity exist here. The first one is the electric singularity, characterized by an electric potential

$$\phi = A\, P_\nu(\cos\Theta)\, R^\nu \tag{2}$$

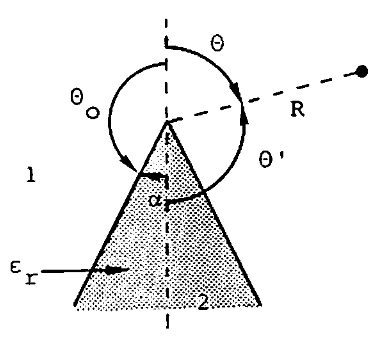

Fig. 5

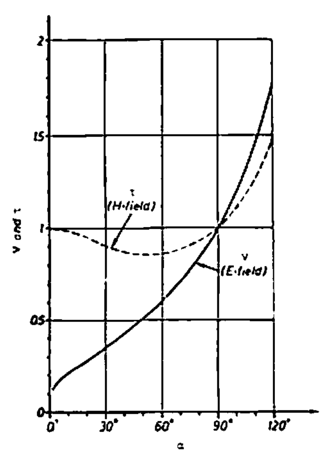

Fig. 6

This potential must vanish for $\Theta = \Theta_o$, a condition which quantizes ν. Values of ν less than one occur for a sharp cone ($\alpha < \pi/2$), and produce an electric field $\bar{e}$ of order $R^{\nu-1}$ (Fig. 6). The charge density ρ_s is of the same order. The second singularity is magnetic, and is characterized by a potential

$$\psi = B\, P_\tau^1(\cos\Theta)\cos(\varphi-\varphi_o)R^\tau \tag{3}$$

The boundary condition $(\partial\psi/\partial\Theta) = 0$ at $\Theta = \Theta_o$ quantizes τ, the lowest value of which is shown in Fig. 6. It is $\bar{h}$ which is singular now, together with the surface current density $\bar{j}_s$ near the tip [5].

Potentials of the general form (2) can be used to investigate the singularities at the tip of dielectric and magnetic cones. For a sharp cone the singularity is associated with a φ-independent potential, and the exponent is in Fig. 7. For a reentrant cone the potential is of the $\cos\varphi$ form, and the exponent is in Fig. 8 [6].

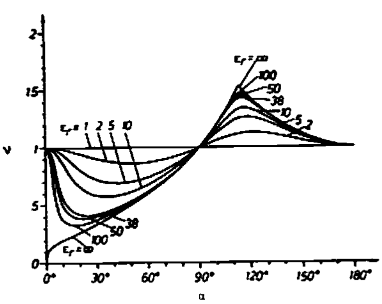

Fig. 7

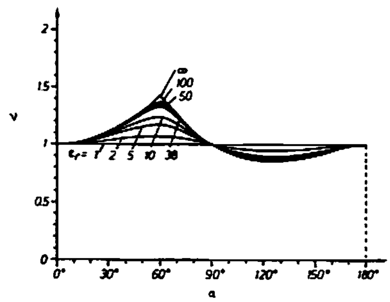

Fig. 8

To determine the singularities at the tip of a metallic cone of arbitrary cross section we assume a potential of the general form $\phi = A\, Y_{mn}(\Theta,\varphi)R^{\nu_{mn}}$, and solve the eigenvalue problem

$$\nabla^2_{\Theta\varphi} Y_{mn} + \nu_{mn}(\nu_{mn}+1)Y_{mn} = 0 \tag{4}$$

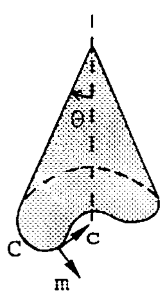

Fig. 9

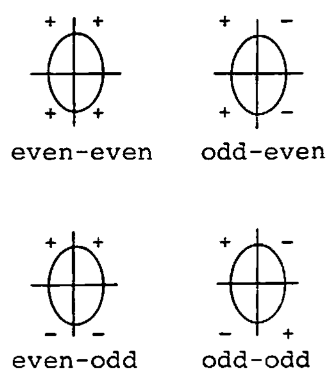

Fig. 10

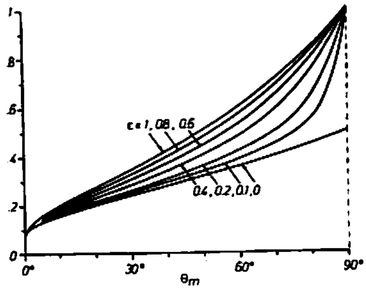

Fig. 11

The electric singularity is obtained by applying the boundary condition Y = 0 on (c), where c is the intersection of the cone with a sphere of radius R = 1. The magnetic singularity corresponds to the boundary condition $(\partial Y/\partial m) = 0$ on (c). We have solved the eigenvalue problem for a series of cone shapes, e.g. the pyramid and the elliptic cone [7]. A few results are shown for the elliptic cone of arbitrary eccentricity ε, where ε = 1 corresponds to the circular cone and ε = 0 to the sector.

The only electric singularity occurs for the even-even symmetry (Fig. 10), for which the relevant ν is shown in Fig. 11, where Θ_m is the maximum half-opening angle (obtained at the apex of the major axis of the ellipse). For the magnetic singularity, both odd-even and even-odd symmetries give rise to singularities. Their respective singularity exponents τ are shown in Figs. 12 and 13. Only the odd-even singularity subsists for the sector.

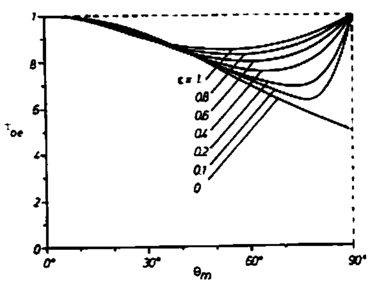

Fig. 12

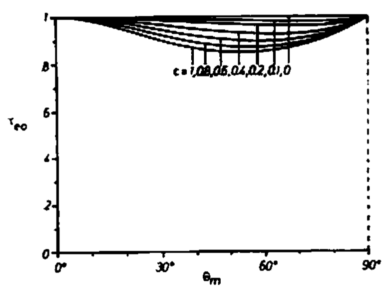

Fig. 13

REFERENCES

[1] J. Meixner, "The behavior of electromagnetic fields at edges",IEEE Trans. AP-20, 442-446, 1972.
[2] J. Bach Andersen, "Field behavior near a dielectric wedge",IEEE Trans. AP-26, 598-602, 1978.
[3] J. Van Bladel, "Induction at a corner in perfect iron", IEE. Proc. 128-B, 219-221, 1981.
[4] J. Van Bladel, "Field singularities at metal-dielectric wedges", to appear shortly in IEEE Trans. AP.
[5] J. Van Bladel, "Field singularities at the tip of a cone", Proc. IEEE, 71, 901-902, 1983.
[6] J. Van Bladel, "Field singularities at the tip of a dielectric cone", accepted for publication in IEEE Trans. AP.
[7] R. De Smedt and J. Van Bladel, "Field singularities at the tip of cones of arbitrary cross-section", Internal Report, Laboratory for Electromagnetism and Acoustics, march 1985.