RELATION BETWEEN CONVENTIONAL AND COMPLEX-ARGUMENT HERMITE-GAUSSIAN BEAMS AND ITS APPLICATION TO SCATTERING BY CYLINDERS

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INTRODUCTION

As is well known, the laser beams with small divergence angle are approximated well by the conventional real-argument Hermite-Gaussian beams [1]. These beams are often used as a basis set to analyze the propagation and diffraction problems of light beams. There is an alternative set of Hermite-Gaussian functions of complex argument proposed by Siegman [2]. As far as we know, the relation between the real-argument and the complex-argument Hermite-Gaussian beams has not been shown.

In this paper, we give the relation between two basis sets. Since it has been shown that the complex-argument Hermite-Gaussian beams are generated by multipoles with complex source points, the conventional beams are expressed as a superposition of multipole fields with complex source points in the paraxial region. As an application of the expressions of the conventional beams, scattering of beams by parallel cylinders is examined. For simplicity, we consider two-dimensional sources and scattering by two cylinders. The time factor $\exp(j\omega t)$ is suppressed.

CONVENTIONAL BEAMS GENERATED BY MULTIPOLES WITH COMPLEX SOURCE POINTS

The conventional Hermite-Gaussian beams $\psi_n(x,z)$ propagating along the z direction are represented by [1]

$$\psi_n(x,z) = \left(\frac{2}{\pi}\right)^{1/4}\left(\frac{1}{n!w_0 2^n}\right)^{1/2}\sqrt{\sigma}\, H_n\left(\sqrt{2}|\sigma|\frac{x}{w_0}\right)\exp\left[-\sigma\left(\frac{x}{w_0}\right)^2+jn\arg\sigma - jkz\right], \tag{1}$$

where k is the wave number in free space and

$$\sigma = [1-j(z/b)]^{-1}, \quad b = kw_0^2/2. \tag{2}$$

The lowest order Gaussian beam given by Eq.(1) has the smallest spot size w_0 at $z = 0$. The complex-argument Hermite-Gaussian beams proposed by Siegman are expressed as [2]

$$\hat{\psi}_n(x,z) = \sigma^{(n+1)/2} H_n\left(\sqrt{\sigma}\frac{x}{w_0}\right)\exp\left[-\sigma\left(\frac{x}{w_0}\right)^2 - jkz\right]. \tag{3}$$

Using the relation for the Hermite polynomials [3]

$$H_n(\kappa\xi) = \sum_{q=0}^{[n/2]} \frac{n!}{q!(n-2q)!}\kappa^{n-2q}(\kappa^2-1)^q H_{n-2q}(\xi), \tag{4}$$

we obtain the relation between ψ_n and $\hat{\psi}_n$ as follows:

$$\psi_n = \sum_{q=0}^{[n/2]} \left(\frac{2}{\pi}\right)^{1/4}\left(\frac{n!}{w_0}\right)^{1/2} \frac{2^{-q}}{q!(n-2q)!}\hat{\psi}_{n-2q}, \tag{5}$$

$$\hat{\psi}_n = \sum_{q=0}^{[n/2]} \left(\frac{\pi}{2}\right)^{1/4}\left(\frac{w_0}{(n-2q)!}\right)^{1/2} \frac{(-2)^{-q}n!}{q!}\psi_{n-2q}. \tag{6}$$

Following Shin and Felsen [4], we define the fields excited by the multipoles located at a complex point (0, -jb) as

$$G_n(x,z) = \frac{\partial^n}{\partial x^n} H_0^{(2)}(kR), \tag{7}$$

where the complex distance R from the source point to a field point is $\sqrt{x^2 + (z + jb)^2}$. If the field point is far from the branch points ($k|R| \gg 1$) and in the paraxial region ($x^2 \ll z^2 + b^2$), the multipole fields are approximated by

$$G_n \sim (-1)^n \pi^{-1/2} \frac{2}{k} w_0^{-(n+1)} e^{kb} \hat{\psi}_n. \tag{8}$$

Using Eqs.(5) and (8) leads to the expressions of the conventional beams in terms of the multipole fields, namely

$$\psi \sim (-1)^n k \left(\frac{\pi}{2}\right)^{1/4} \left(\frac{n! w_0}{2}\right)^{1/2} e^{-kb} \sum_{q=0}^{[n/2]} \frac{2^{-q}}{q!(n-2q)!} w_0^{n-2q} G_{n-2q}. \tag{9}$$

SCATTERING BY TWO CYLINDERS

Let a beam expressed as Eq.(1) where x and z are replaced by x′ and z′ be incident on two parallel cylinders #1 with radius a^1 and refractive index n^1 located at (0, d/2), and #2 with radius a^2 and index n^2 at (0, -d/2) as shown in Fig.1. The angle φ_0 is the angle between the beam and the z axes and (x_0, $-z_0$) is the location of the beam waist. Scattering of beams whose electric fields are parallel to the axes of the cylinders is analyzed.

First, we investigate the scattered fields for the multipole fields G_n, given by Eq.(7) where x and z are replaced by x′ and z′, incident on the cylinders. Making use of the additional theorem for the Bessel functions leads to the expressions of the multipole fields G_n in the co-ordinate systems (r^i, θ^i) as

$$G_n = \begin{cases} \displaystyle\sum_{m=-\infty}^{\infty} \alpha_m^i(n)\, H_m^{(2)}(kr^i)\exp(jm\theta^i) , & |\rho^i| < r^i , \quad \text{(10a)} \\ \displaystyle\sum_{m=-\infty}^{\infty} \beta_m^i(n)\, J_m(kr^i) \exp(jm\theta^i) , & |\rho^i| > r^i, \quad i = 1,2, \quad \text{(10b)} \end{cases}$$

where, unlike the recurrence relations for the coefficients of the three-dimensional multipole fields [5], $\alpha_m^i(n)$ and $\beta_m^i(n)$ are given explicitly by

$$\begin{Bmatrix} \alpha_m^i(n) \\ \beta_m^i(n) \end{Bmatrix} = (j\frac{k}{2})^n \sum_{q=0}^{n} \binom{n}{q} \begin{Bmatrix} J_{m+n-2q}(k\rho^i) \\ H_{m+n-2q}^{(2)}(k\rho^i) \end{Bmatrix} \exp[-j(m+n-2q)\phi^i - j(n-2q)\varphi_0] \tag{11}$$

and where

$$r^i = \sqrt{(x \mp d/2)^2 + z^2} , \qquad \theta^i = \tan^{-1}[(x \mp d/2) / z],$$

$$\rho^i = \sqrt{(x_0 + jb\sin\varphi_0 \mp d/2)^2 + (z_0 + jb\cos\varphi_0)^2} ,$$

$$\phi^i = \tan^{-1}[-(x + jb\sin\varphi_0 \mp d/2) / (z + jb\cos\varphi_0)], \tag{12}$$

with ∓ signs corresponding to i = 1 and 2, respectively. The scattered fields $e_n^{s(i)}$ by the cylinder #i are expressed as

$$e_n^{s(i)} = \sum_{m=-\infty}^{\infty} C_m^i(n)\, H_m^{(2)}(kr^i)\exp(jm\theta^i). \tag{13}$$

The fields $e_n^{t(i)}$ in the cylinder #i are expressed as

$$e_n^{t(i)} = \sum_{m=-\infty}^{\infty} D_m^i(n) J_m(n^i k r^i) \exp(jm\theta^i). \tag{14}$$

Applying the boundary conditions at $r^i = a^i$ after expressing the scattered fields $e_n^{s(1)}$ and $e_n^{s(2)}$ in the co-ordinate systems (r^2, θ^2) and (r^1, θ^1), respectively, we obtain the equations which determine the unknown coefficients $C_m^i(n)$ and $D_m^i(n)$. The result is

$$\begin{aligned}
C_m^1(n) &= A_m^1 + D_m^1(n)F_m^1 + \sum_{p=-\infty}^{\infty} C_p^2(n)B_{mp}^1, \\
C_m^2(n) &= A_m^2 + D_m^2(n)F_m^2 + \sum C_p^1(n)B_{mp}^2, \\
D_m^1(n) &= \bar{A}_m^1 + C_m^1(n)\bar{F}_m^1 + \sum C_p^2(n)\bar{B}_{mp}^1, \\
D_m^2(n) &= \bar{A}_m^2 + C_m^2(n)\bar{F}_m^2 + \sum C_p^1(n)\bar{B}_{mp}^2, \quad m = 0, \pm 1, \cdots,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
A_m^i &= -\beta_m^i(n) J_m(ka^i)/H_m^{(2)}(ka^i), \quad \bar{A}_m^i = \beta_m^i(n) J_m'(ka^i)/n^i J_m'(n^i ka^i), \\
B_{mp}^i &= -(\mp j)^{m-p} H_{p-m}^{(2)}(kd) J_m(ka^i)/H_n^{(2)}(ka^i), \\
\bar{B}_{mp}^i &= (\mp j)^{m-p} H_{p-m}^{(2)}(kd) J_m'(ka^i)/n^i J_m'(n^i ka^i), \\
F_m^i &= J_m(n^i ka^i)/H_m^{(2)}(ka^i), \quad \bar{F}_m^i = H_m^{(2)\prime}(ka^i)/n^i J_m'(n^i ka^i).
\end{aligned} \tag{16}$$

Since the beams are expressed in terms of the multipole fields, the total scattered fields E_n^s for the incident beams ψ_n are given by a superposition of the scattered fields $e_n^{s(i)}$ as follows:

$$E_n^s \sim (-1)^n k \left(\frac{\pi}{2}\right)^{1/4} \left(\frac{n! w_0}{2}\right)^{1/2} e^{-kb} \sum_{q=0}^{[n/2]} \frac{2^{-q}}{q!(n-2q)!} w_0^{n-2q} [e_{n-2q}^{s(1)} + e_{n-2q}^{s(2)}]. \tag{17}$$

Using the asymptotic formulas for the Hankel functions in Eq.(17), we obtain the far scattered field.

Numerical calculations have been carried out for the beam with $kw_0 = 4\pi$, $kx_0 = 0$, $kz_0 = 2\pi$ and $\varphi_0 = 0°$, and two equal cylinders with $ka^1 = ka^2 = \pi$ and $kd = 10\pi$. The normalized scattered-field patterns of E_0^s and E_2^s are shown in Figs. 2 and 3, respectively. In both figures, the solid and the dashed curves correspond to scattering by dielectric cylinders with $n^1 = n^2 = 1.5$, and by perfectly conducting cylinders, respectively. The scattered-field pattern of E_0^s by conducting cylinders shown in Fig. 2 agrees well with that given by Kojima et al.[6].

CONCLUSIONS

The relation between the conventional and the complex-argument Hermite-Gaussian beams have been given. Using the relation, the conventional beams are expressed in terms of the fields excited by the multipoles with complex source points in the paraxial region. Scattering of beams by two cylinders has been considered as an application of the expressions. The method used in this paper is useful for analyzing focusing of laser beams by a cylindrical lens and can be also extended to scattering of arbitrary configuration of parallel cylinders.

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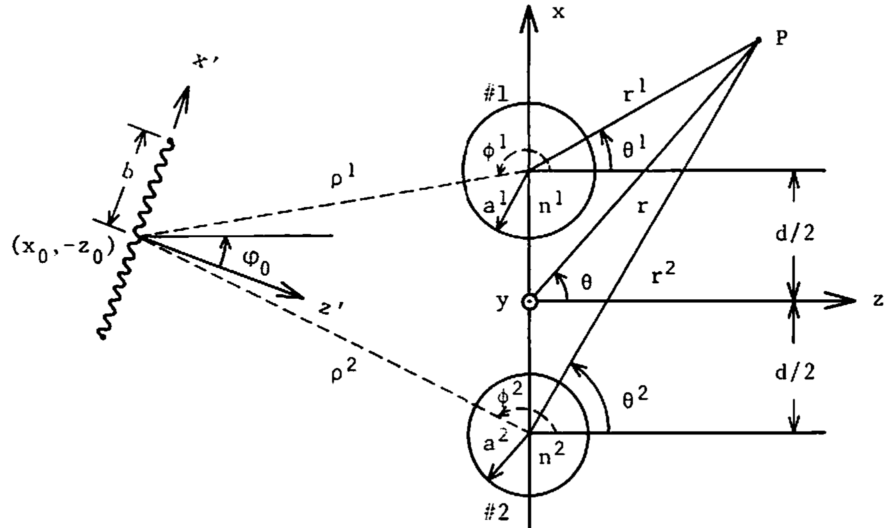

Fig. 1. Geometry of the problem.

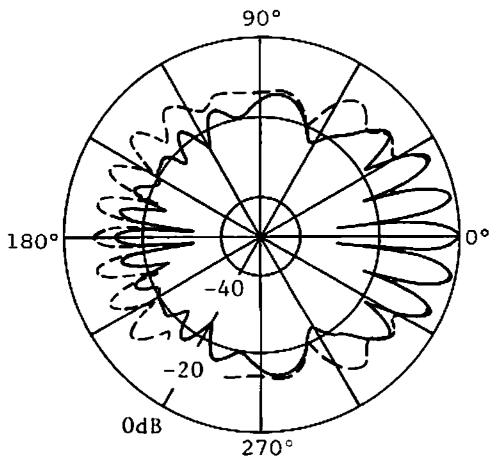

Fig. 2. Radiation patterns of E_0^s.
—— by dielectric cylinders.
---- by conducting cylinders.

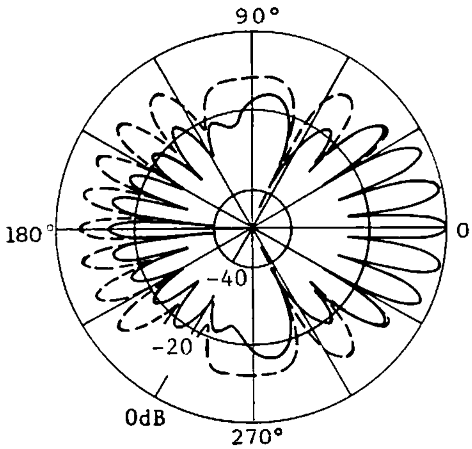

Fig. 3. Radiation patterns of E_2^s.
—— by dielectric cylinders.
---- by conducting cylinders.