B–8–4

NUMERICAL METHOD FOR SOLVING EDGE-TYPE SCATTERING PROBLEMS
— SINGULAR-SMOOTHING PROCEDURE ON MODE-MATCHING METHOD —

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1. Introduction

The mode-matching method based on the Rayleigh expansion theorem [1] is a general procedure for solving boundary value problems on the electromagnetic fields. According to the Rayleigh theorem, the truncated modal expansion is made to fit to the boundary condition in the sense of least squares, and it is guaranteed that the sequence of the truncated modal expansion converges to the true solution of the problem as the number of truncation tends to infinity. This method, theoretically, can be applied to problems with arbitrarily shaped boundaries. However, in numerical calculations, the convergence is not so fast as to analyse problems with complicated boundaries in high accuracy. In order to overcome this difficulty, the smoothing procedure [2] was proposed. This procedure indeed improves the conventional mode-matching method when the boundary of the scatterer is sufficiently smooth. But when the boundary has an edge-point, one cannot use this procedure without any considerations on the singularity due to the existance of the edge-point.
In this paper, we will introduce singular-smoothing procedure (SSP) [3] and show the usefulness of this new procedure for solving edge-type scattering problems. The time factor $\exp(i\omega t)$ is supressed throughout.

2. Singular-smoothing procedure on mode-matching method

A cylindrical scatterer made of a perfect electric conductor whose cross section is shown in Fig.1 will be considered. In this figure, A denotes the edge-point on a closed contour L which shows the shape of the scatterer. The point P with Cartesian co-ordinates (x,y) is in the exterior infinite domain S, and a point on L is signified by an arc-length s measured from A.

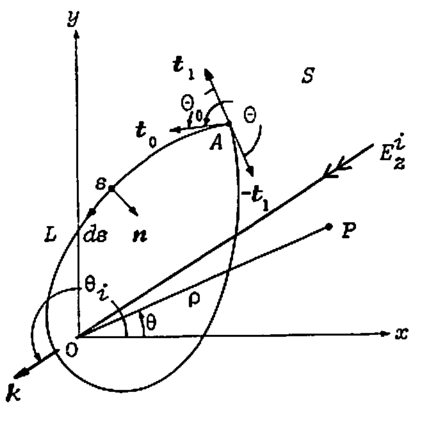

Fig.1 The cross section of an edge-type scatterer

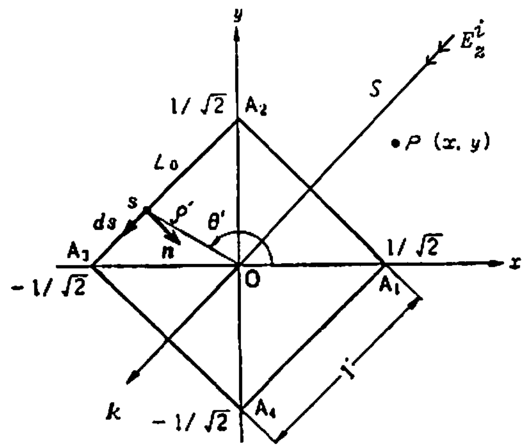

Fig.2 Cross-sectional view of a quadrate cylinder

The arc-length s is normalized by the total length of L, therefore $s=0$ and $s=1$ correspond to the edge-point A.
When the cylinder is illuminated by an E-polarized plane wave

$$E_z^i(P) = \exp(-ik(x\cos\theta_i+y\sin\theta_i)), \tag{1}$$

the scattered field $E_z^s(P)$ becomes a diverging wave function in S and satisfies Dirichlet's condition

$$E_z^s(s) = -E_z^i(s) \tag{2}$$

on the boundary L.
Let ϕ_m's

$$\phi_m(P) = H_m^{(2)}(k\rho)\exp(im\theta) \qquad m=0,\pm1,\pm2,\cdots \tag{3}$$

be a set of modal functions. A truncated modal expansion is constructed as

$$E_{zN}(P) = \sum_{m=-N}^{N} a_m(N)\phi_m(P), \tag{4}$$

where N is the number of truncation.
The wave function $E_z^s(P)$ can be represented by an integral form on L

$$E_z^s(P) = \int_0^1 E_z^i(s)\ \partial G(P,s)/\partial n\ ds \tag{5}$$

where $G(P,Q)$ is the Green function for this Dirichlet problem, and $\partial/\partial n$ denotes normal derivative on L. The truncated modal expansion $E_{zN}(P)$ can be represented in the same form. Subtructing these equations, the relation

$$E_z^s(P)-E_{zN}(P) = \int_0^1 \delta e(s,N)\ \partial G(P,s)/\partial n\ ds \tag{6}$$

where

$$\delta e(s,N) = E_z^i(s) + E_{zN}(s)$$

is derived. This relation is the fundamental expression for the mode-matching technique, though it is only a formal one. It should be noted, as one can easily see, that the normal derivative of the Green function asymptotically has the form

$$\partial G(P,s)/\partial n \sim s^{\sigma-1}\ (s\simeq0), \quad \sim (1-s)^{\sigma-1}\ (s\simeq1), \tag{7}$$

where σ is a parameter indicating the sharpness of the edge and is defined

$$\sigma = \pi/\Theta \text{ (see Fig.1)}. \tag{8}$$

The parameter σ is always greater than 1/2. For the sake of simplicity, we restrict our discussion within the case of $1/2 < \sigma < 1$.
As σ is greater than 1/2, one may apply Schwarz'inequality to eq.(6) and obtain

$$|\ E_z^s(P)-E_{zN}(P)\ | \le K^0 \|\ \delta e(N)\ \| \qquad (\ P\in \tilde{S}\ ) \tag{9}$$

as long as P belongs to any subdomain $\tilde{S}$ in the exterior domain S. Here, K^0 is a positive constant which is independent of P or N. The norm $\|\cdot\|$ is defined as

$$\|\ \delta e(N)\ \| = \left(\ \int_0^1 |\delta e(s,N)|^2 ds\ \right)^{1/2}. \tag{10}$$

A numerical technique based on the conventional mode-matching method is formulated so as to minimize the norm (10).

Next, we introduce the singular-smoothing procedure. If the constraint

$$\int_0^1 \delta e(s,N)\ ds = 0 \tag{11}$$

is held, eq.(6) can be transformed into

$$E_z^s(P)-E_{zN}(P) = -\int_0^1 [\ \mathbf{H}\delta e(s,N)/w(s)\ ]\ [\ w(s)\partial^2 G(P,s)/\partial n\partial s\ ]\ ds \tag{12}$$

through integration by parts, where

$$\mathrm{H}\delta e(s,N) = \int_0^s \delta e(t,N)\,dt, \qquad w(s) = s(1-s). \tag{13}$$

Applying Schwarz'inequality to eq.(12), we get the estimation

$$|\ E_z^s(P)-E_{zN}(P)\ | \le K^1 \|\ \mathrm{H}\delta e(N)/w\ \| \qquad (\ P\in \tilde{S}\ ). \tag{14}$$

With the aid of some considerations based on the theory of functional analysis, we can prove the following lemma:

Lemma 1. For any boundary value $E_z^i(s)$, there exists an infinite sequence of expansion coefficients for the truncated modal expansion defined by eq.(4)

$$\{a_0(0)\},\{a_m(1);m=0,\pm1\},\cdots\{a_m(N);m=0,\pm1,\pm2,\cdots,\pm N\},\cdots \tag{15}$$

such that the norm in r.h.s. of (14) vanishes as N tends to infinity;

$$\lim_{N\to\infty} \|\ \mathrm{H}\delta e(N)/w\ \| = 0. \tag{16}$$

The algorithm for determining the expansion coefficients (15) is given by minimization of the norm in r.h.s. of (14) under the constraint (11). The expression of the norm can be simplified as

$$\|\ \mathrm{H}\delta e(N)/w\ \|^2 = \int_0^1\int_0^1 W_s(s,s')\ \overline{\delta e(s,N)}\ \delta e(s',N)\ ds\ ds' \tag{17}$$

where $\overline{\delta e}$ is the complex conjugate quantity of δe, and $W_s(s,s')$ is a symmetric kernel function with respect to s and s':

$$W_s(s,s') = \log\frac{s'(1-s)}{s(1-s')} + \frac{1}{s} + \frac{1}{1-s'} \qquad (\ 0 < s' < s < 1\ ) \tag{18}$$

Thus the sequence of the truncated modal expansion does converge uniformly to the true solution in $\tilde{S}$. It is also reduced that the modal expansion obtained by this algorithm converges more rapidly than one determined by means of the conventional mode-matching method.

3. Numerical analysis of scattering from a quadrate cylinder

Fig.2 shows the cross-sectional view of a quadrate cylinder. Its scattering properties are analysed, and some results are shown in Figs.3,4.
The accuracy of numerical solutions calculated by above two methods is shown in Fig.3; decrement of errors on the optical theorem is drawn as function of truncated number N for k=1 and 15. The rapid decrement of the solid line (mode-matching method using SSP) suggests that SSP is a useful technique for solving edge-type scattering problems.
Fig.4 displays a numerical example; the back scattering cross section versus the wave number k is described for some different incident angles.

4. Conclusion

A computer-aided numerical technique for solving edge-type scattering problems is proposed, and its efficiency is demonstrated.
This method can be applied to other kinds of edge-type problems [4], for example, the Neumann-type boundary value problems.

References

[1] K.Yasuura,"A view of numerical methods in diffraction problems", The XVI-th General Assembly of URSI,Ottawa(1969).
[2] ———,"Smoothing process on mode-matching technique for solving Helmholtz equation",URSI Symposium on electromagnetic wave theory, London(1974).

[3] K.Yasuura and Y.Okuno,"Singular-smoothing process on series expansion of functions",Tech.Rep.Kyushu univ.,51,1(1978).
[4] ———,"Algorithm for solving edge-type reflection gratings",to be published on Trans.IECE,J61-B(1978).

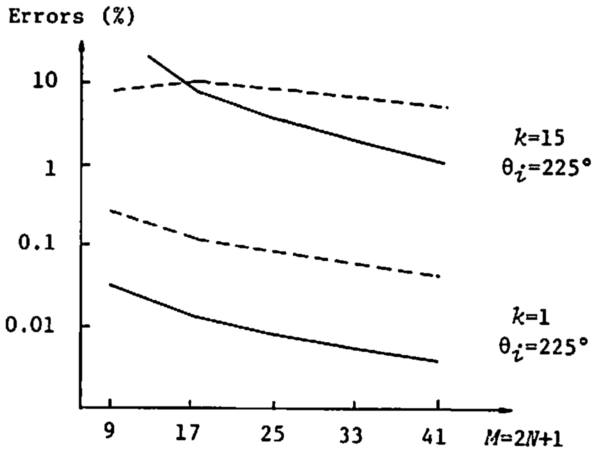

Fig.3 Errors on the optical theorem versus the number of truncation; the solid line and the dashed line correspond to the mode-matching method using SSP and the conventional one respectively.

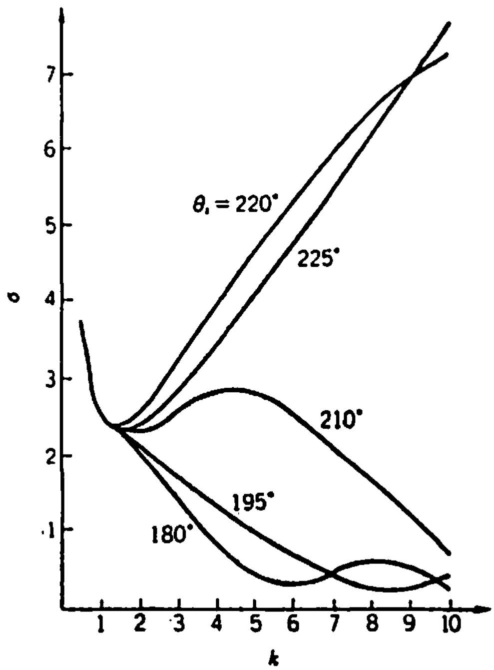

Fig.4 Back scattering cross section versus wave number k.