A NOVEL PERTURBATION ANALYSIS OF COUPLED OPTICAL WAVEGUIDES

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1. Introduction

Perturbation technique is one of the most commonly used approximate approaches for physical problems, because perturbation expansions are useful for a qualitative as well as a quantitative representation of the solution[1]. In the analysis of optical waveguide, the characteristics of several components such as directional couplers[2] and grating couplers[3], which are regarded as perturbed waveguides, have studied by using this approach. In this approach, the isolated waveguides without any disturbance are used as the unperturbed waveguide. However, the equations governing electromagnetic fields in the unperturbed waveguides are usually described in terms of various physical parameters with different magnitude. This makes it difficult to evaluate the perturbation effects in adequate manner, both in the propagation and transverse directions of the waveguides.

In this paper, we consider the scalar Helmholtz equation describing the electromagnetic fields in dielectric waveguide as an unperturbed equation. We introduce slowly varying amplitude representation in order to rearrange the unperturbed equation. This scheme enables us to evaluate the magnitude of applied perturbation effects in proper manner. We apply this scheme to the analysis of a directional coupler made of two nonidentical dielectric slab waveguides by using the singular perturbation technique[2]. It is shown that propagation constants of the coupled-modes obtained by the perturbation analysis are in very close agreement with the exact ones[4].

2. Formulation

We consider a guided wave propagating along an unperturbed dielectric waveguide in which electromagnetic fields can be described by the scalar Helmholtz equation. Let us take the z axis along the direction of propagation, then the wave function $\Psi(x, y, z)$ for the total field satisfies the scalar Helmholtz equation:

$$\left[\nabla_t^2 + \frac{\partial^2}{\partial z^2} + k_s^2\left(1 + \Delta\varepsilon\, w(x, y)\right)\right] \Psi(x, y, z) = 0, \tag{1}$$

where ∇_t^2 denotes the transverse Laplacian, k_s is the wavenumber in the cladding, $\Delta\varepsilon$ is the relative permittivity difference between the core and the cladding, $w(x, y)$ is the window function which has nonzero value only in the core region. In optical waveguides, variation of the field in the z direction is generally much more rapid than in the transverse direction. Then, comparing the magnitudes of the respective terms involved in Eq. (1), the first and fourth terms on the left side are much smaller than the second and third terms. Since the first and fourth terms are related to the transverse variations of the wave field, the solution to the wave equation is much more sensitive to the perturbations in the transverse direction than those in the z direction.

To avoid this difficulty, we need to rearrange the unperturbed wave equation. However, when the first and fourth terms on the left side in Eq. (1) are treated as perturbation terms, the highest partial derivative in the transverse direction degenerates and hence the unperturbed equation does not lead to the wave function of the problem. Therefore we introduce the slowly varying amplitude in order to reduce the magnitude of the partial derivative in the z direction. When the wavenumber in the cladding is chosen as the reference propagation constant, the slowly varying amplitude is defined as

$$\psi(x, y, z) = \Psi(x, y, z)\, e^{-jk_s z}. \tag{2}$$

Using this function, one may obtain the following equation from Eq. (1):

$$\left(\nabla_t^2 + \frac{\partial^2}{\partial z^2} - j\,2\,k_s \frac{\partial}{\partial z} + k_s^2\, \Delta\varepsilon\, w(x, y)\right) \psi(x, y, z) = 0. \tag{3}$$

Comparing the magnitudes of the terms on the left side in Eq. (3) with $|\nabla_t^2\, \psi| = \Delta\varepsilon\, k_s\, |\psi| / 2$, the second term is much smaller than the other three terms, which are the same order in magnitude. Therefore, if the second term is treated as a perturbation term, we can obtain the unperturbed wave equation with the terms of the same order in magnitude. We note that the highest derivative of the differential equation in the z direction also degenerates in the unperturbed problem. However, when the waves propagating in the positive z direction are concerned, this degeneracy causes no trouble.

3. Asymmetric Directional Coupler

In this section, we consider an asymmetric coupler shown in Fig. 1 in order to confirm the present scheme. It consists of two nonidentical dielectric slab waveguides a and b with thicknesses $2\,d_a$ and $2\,d_b$, and permittivity ε_a and ε_b, respectively. They are situated parallel to each other with a separation distance $s_a - s_b - d_a - d_b$ in a surrounding dielectric with permittivity ε_s. The geometry is uniform in the y direction and the two dimensional problem is considered. We shall consider only transverse electric (TE) wave which propagate in the z direction. Each waveguide is supposed to allow only one fundamental mode to propagate and we deal with only the wave propagating in the positive z direction. In our analysis we assume the permeability is uniform and equal to that of vacuum.

The wave function $\Psi(x, z)$ for the total field satisfies the following scalar Helmholtz equation:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_s^2\left(1 + \Delta\varepsilon_a\, w_a(x) + \Delta\varepsilon_b\, w_b(x)\right)\right] \Psi(x, z) = 0, \tag{4}$$

where, for $\nu = a, b$,

$$\Delta\varepsilon_\nu = \frac{n_\nu^2 - n_s^2}{n_s^2}, \quad w_\nu(x) = \begin{cases} 1 & : \ |x - s_\nu| \le d_\nu \\ 0 & : \ |x - s_\nu| > d_\nu \end{cases}. \tag{5}$$

The boundary conditions are given as $\Psi(x, z)$ and its partial derivative of x are continuous for $x = s_\nu \pm d_\nu$. We apply the singular perturbation technique[2] to the analysis. We decompose the wave function $\Psi(x, z)$ and Eq. (4) as follows:

$$\Psi(x, z) = \left(\psi_a(x, z) + \psi_b(x, z)\right) e^{-jk_s z}, \tag{6}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - j\,2\,k_s \frac{\partial}{\partial z} + k_s^2\, \Delta\varepsilon_\nu\, w_\nu(x)\right) \psi_\nu(x, z) = -\,k_s^2\, \Delta\varepsilon_\nu\, w_\nu(x)\, \psi_\mu(x, z), \tag{7}$$

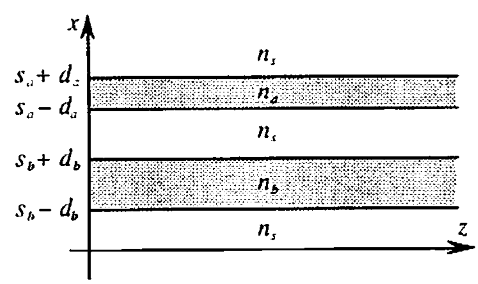

Fig. 1 Geometry of the asymmetric directional coupler made of two nonidentical dielectric slab waveguides

where $\nu, \mu = a, b$, $\nu \neq \mu$, and these notations are used in the following. From the uniqueness theorem, it is verified that the composite wave function $\Psi(x, z)$ satisfies Eq. (4) when the slowly varying amplitudes $\psi_a(x, z)$ and $\psi_b(x, z)$ are solutions to Eq. (7). We treat the second term on the left side and the right side of Eq. (7) as perturbation terms and assume that they are the same order in magnitude. We introduce a nondimensional small parameter δ to identify that those perturbation terms start from the order of δ in magnitude and rewrite Eq. (7) as

$$\left(\frac{\partial^2}{\partial x^2} - j\,2\,k_s \frac{\partial}{\partial z} + k_s^{\,2}\,\Delta\varepsilon_\nu\, w_\nu(x)\right) \psi_\nu(x, z) = -\,\delta\left(\frac{\partial^2}{\partial z^2}\,\psi_\nu(x, z) + k_s^{\,2}\,\Delta\varepsilon_\nu\, w_\nu(x)\,\psi_\mu(x, z)\right). \quad (8)$$

In addition, we introduce the multiple space scales[1]: $z_n = \delta^n z$ (n = 0, 1, ⋯) in order to avoid nonuniformity for large z, and expand the slowly varying amplitudes $\psi_a(x, z)$ and $\psi_b(x, z)$ as

$$\psi_\nu(x, z) = \sum_{n=1}^{\infty} \delta^n\, \psi_{\nu,n}(x, z_{s,n}), \quad (9)$$

where the notation $z_{s,n}$ denotes the series of the space scales $z_n, z_{n+1}, z_{n+2}, \cdots$. Substituting Eq. (9) into Eq. (7), we derive the equations in the respective order δ^n of perturbation by equating the coefficient of each power of δ to zero. The equations up to the second order are obtained as follows:

$$L_{\nu,0}\,\psi_{\nu,0} = 0, \quad (10)$$

$$L_{\nu,0}\,\psi_{\nu,1} = -\,L_1\,\psi_{\nu,0} - M_\nu\,\psi_{\mu,0}, \quad (11)$$

$$L_{\nu,0}\,\psi_{\nu,2} = -\,L_1\,\psi_{\nu,1} - L_2\,\psi_{\nu,0} - M_\nu\,\psi_{\mu,1}, \quad (12)$$

where the linear operators appeared in these equations are defined as follows:

$$L_{\nu,0} \equiv \frac{\partial^2}{\partial x^2} - j\,2\,k_s \frac{\partial}{\partial z_0} + k_s^{\,2}\,\Delta\varepsilon_\nu\, w_\nu(x),\quad L_1 \equiv \frac{\partial^2}{\partial z_0^{\,2}} - j\,2\,k_s \frac{\partial}{\partial z_1}, \quad (13)$$

$$L_2 \equiv 2\left(\frac{\partial^2}{\partial z_0 \partial z_1} - j\,k_s \frac{\partial}{\partial z_2}\right),\quad M_\nu \equiv k_s^{\,2}\,\Delta\varepsilon_\nu\, w_\nu(x). \quad (14)$$

For the zero order problem, the differential equation given by Eq. (10) is homogeneous. Noting that we are dealing with only the guided mode propagating in the positive z direction and each waveguide allows only one fundamental mode, the solution of Eq. (10) can be expressed as

$$\psi_{\nu,0}(x, z_{s,0}) = a_{\nu,0}(z_{s,1})\, u_{\nu,0}(x)\, e^{-j\beta_{\nu,0} z_0}, \quad (15)$$

where $a_{\nu,0}(z_{s,1})$ is the modal amplitude, $u_{\nu,0}(x)$ is the modal profile function, and $\beta_{\nu,0}$ is the zero order correction of the propagation constant.

For the higher order problem, the differential equations given by Eqs. (11) and (12) are inhomogeneous, then the solutions with finite amplitude are allowed only when solvability conditions are satisfied. These solvability conditions for Eqs. (11) and (12) respectively yield the first and second order coupled-mode equations for the modal amplitudes. These coupled-mode equations are combined and the space scales z_1 and z_2 is transformed back into the original space scale z by letting δ = 1. Then, when $(\beta_{a,0} - \beta_{b,0}) / \beta_{a,0}$ is the order of δ in magnitude, the coupled-mode equations for the modal amplitudes $a_{\nu,0}(z)$ with the corrections of the first and second orders of perturbation are derived in self-consistent manner as follows:

$$\frac{\partial}{\partial z}\left(a_{\nu,0}\, e^{-j\beta_{\nu,0} z}\right) = -\,j\left(\beta_{\nu,0} + \beta_{\nu,1}\right) a_{\nu,0}\, e^{-j\beta_{\nu,0} z} - j\,\xi_{\nu\mu,1}\, a_{\mu,0}\, e^{-j\beta_{\mu,0} z}, \quad (16)$$

$$\frac{\partial}{\partial z}\left(a_{\nu,0}\, e^{-j\beta_{\nu,0} z}\right) = -\,j\left(\beta_{\nu,0} + \beta_{\nu,1} + \beta_{\nu,2} + \xi_{\nu,2}\right) a_{\nu,0}\, e^{-j\beta_{\nu,0} z} - j\left(\xi_{\nu\mu,1} + \xi_{\nu\mu,2}\right) a_{\mu,0}\, e^{-j\beta_{\mu,0} z}, \quad (17)$$

where

$$\beta_{\nu,1} = -\,\frac{\beta_{\nu,0}^{\;2}}{2\,k_s},\quad \xi_{\nu\mu,1} = \frac{1}{2}\,k_s\,\Delta\varepsilon_\nu \int_{s_\nu - d_\nu}^{s_\nu + d_\nu} u_{\nu,0}\, u_{\mu,0}\, dx, \quad (18)$$

$$\beta_{v,2} = -\frac{\beta_{v,0}\,\beta_{v,1}}{k_s},\ \xi_{v,2} = \frac{1}{2}\,k_s\,\Delta\varepsilon_v \int_{s_v-d_v}^{s_v+d_v} u_{v,0}\,u_{\mu v,1}\,dx,\ \xi_{v\mu,2} = -\frac{\beta_{v,0}\,\xi_{v\mu,1}}{k_s}, \tag{19}$$

and $u_{v\mu,1}$ is the first order correction of the modal profile function given by the particular solution to the first order equation (11), which solution is chosen to be orthogonal to $u_{v,0}(x)$. We can easily calculate the coupling coefficients defined in Eqs. (18) and (19).

4. Numerical Example

In order to confirm numerically the accuracy of the present formulation, we have considered TE guided modes of an asymmetric directional coupler with n_s = 3.0, n_a = 3.295, n_b = 3.3, d_a = d_b = 0.2 μm, $s_a - s_b$ = 1.4 μm, and the wavelength λ = 1.5 μm. In this coupler, the perturbation terms involved on the right side of Eq. (8) are several percent in magnitude of the other terms, and $(\beta_{a,0} - \beta_{b,0}) / \beta_{a,0}$ is also the same order in magnitude. We can obtain the approximate propagation constants of the coupled-modes by adding the eigenvalues of the coupled-mode equation (16) or (17) to k_s. Table 1 shows the error rates of the approximate propagation constants as compared with those of exact theory[4]. It is observed that our results are in very good agreement with the exact ones and that the accuracy increases by improving the approximations. Moreover, as the order of the corrections of perturbations increases, the differences decrease with nearly constant ratio which is given by the order of δ in magnitude.

Table 1 Error rates of the approximate propagation constants of TE mode in an asymmetric directional coupler with n_s = 3.0, n_a = 3.295, n_b = 3.3, d_a = d_b = 0.2 μm, and $s_a - s_b$ = 1.4 μm for free space wavelength of 1.5 μm.

mode	exact	error rate			
	(m^{-1})	k_s	$O(\delta^0)$	$O(\delta)$	$O(\delta^2)$
even	1.3231263×10^7	5.03×10^{-3}	1.20×10^{-4}	4.97×10^{-5}	3.92×10^{-6}
odd	1.3211700×10^7	4.88×10^{-3}	1.40×10^{-4}	7.07×10^{-5}	2.95×10^{-6}

5. Conclusion

In this paper, we have investigated the singular perturbation technique for perturbed optical waveguides by using slowly varying amplitude representation. This technique enables us to evaluate the magnitude of applied perturbation effects in proper manner, although the backward propagating waves are not taken into account. We applied this technique to the analysis of an asymmetric directional coupler composed of two nonidentical dielectric slab waveguides. The results are in very good agreement with the exact ones, and the error rates decrease according to the successive orders of perturbation. This fact suggests that the present technique is asymptotically correct for the waveguide problem with perturbations which have directional dependence.

References

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