A THEORY OF MUTUAL IMPEDANCES AND MULTIPLE REFLECTIONS IN AN N-ELEMENT ARRAY ENVIRONMENT

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An N-element array is a complex electromagnetic environment where each element interacts with all other elements and multiple reflections take place between any two subgroups of elements. The total interaction between any two elements in an N-element array can be described by a sum of finite number of terms t_N, recursively given by $t_N = 1 + (N-2)t_{N-1}$, where the first term represents the direct interaction between the excited element and the element under observation, and each of the other terms give part of the environmental effect. To derive expressions for mutual impedances between any two elements in an open-circuited array environment, we start with the scattering matrix formalism [1] and write (considering the reflectionless case, i.e., $S_{\alpha\alpha} = 0$)

$$\begin{pmatrix} \frac{1}{2}(v_\ell - i_\ell) \\ \frac{1}{2}(\underline{V}_\ell - \underline{I}_\ell) \end{pmatrix} = \begin{pmatrix} 0 & \underline{S}^{(1)\dagger}_{\alpha\beta} \\ \underline{S}^{(1)}_{\beta\alpha} & S^{(1)}_0 - \underline{S}^{(1)}_{\beta\alpha}\underline{S}^{(1)\dagger}_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \frac{1}{2}(v_\ell + i_\ell) \\ \frac{1}{2}(\underline{V}_\ell + \underline{I}_\ell) \end{pmatrix} \tag{1}$$

for $\ell=1,N$, where $i_k \neq 0$ if the k-th element is excited, and $i_\ell=0$ for all $\ell \neq k$, since they are open circuited. In addition to (1), $\underline{V}_\ell$ and $\underline{I}_\ell$ are related by mutual impedance matrices $Z_{\ell k}$ [2]. Thus,

$$\begin{pmatrix} \underline{V}_1 \\ \vdots \\ \underline{V}_\ell \\ \vdots \\ \underline{V}_N \end{pmatrix} = \begin{pmatrix} 1 & Z_{12} & \cdots & Z_{1\ell} & \cdots & Z_{1N} \\ \vdots & & \vdots & & & \vdots \\ Z_{\ell 1} & Z_{\ell 2} & \cdots & 1 & \cdots & Z_{\ell N} \\ \vdots & & \vdots & & & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{N\ell} & \cdots & 1 \end{pmatrix} \begin{pmatrix} -\underline{I}_1 \\ \vdots \\ -\underline{I}_\ell \\ \vdots \\ -\underline{I}_N \end{pmatrix} \tag{2}$$

where the elements of $Z_{k\ell}$ are given by the free-space mode-mode mutual impedance integrals [2]

$$\zeta^{(k\ell)}_{\alpha\beta} = 2 \int_0^{2\pi} d\phi \int_{-j\infty}^{1} e^{-jkD\cos\theta}\, \underline{f}^{(k)}_\alpha \cdot \underline{f}^{*(\ell)}_\beta \, d(\cos\theta) \tag{3}$$

where $\underline{f}^{(k)}_\alpha$ is the far field pattern of mode α (corresponding to some set of mode numbers nm) radiated by antenna k. We wish to solve the system of equations (1) and (2) for the mutual impedances z_{ij} between element i and j, defined as [2]

$$z_{ij} = \left.\frac{v_i}{i_j}\right|_{i_k=0,\ k\neq j}, \tag{4}$$

in terms of free space binary current-current interaction matrices defined as

$$\Lambda_{ij} \equiv \frac{1}{2}\,(1 - S_o^{(1)})Z_{ij} \tag{5}$$

where S_o is the scattering matrix of the open circuited antenna [2]. These Λ matrices naturally arise in the course of manipulating the system of equations (1) and (2) to derive the linear system (assuming element k is radiating)

$$\underline{I}_k = -(\Lambda_{k2}\underline{I}_2 + \ . \ . \ . + \Lambda_{kN}\underline{I}_N) - S_{\beta\alpha}^{(k)} i_k \tag{6}$$

$$\underline{I}_\ell = -(\Lambda_{\ell 1}\underline{I}_1 + \ . \ . \ . + \Lambda_{\ell N}\underline{I}_N)'\,, \quad \ell=1,N;\ \ell\neq k \tag{7}$$

with the $\underline{I}_\ell$ term missing on the right side as indicated by the prime. In matrix notation equation (7) is

$$\Lambda\underline{I} = -\hat{\underline{I}}_k. \tag{8}$$

The solution to this linear system can be written as

$$\underline{I}_\ell = -\Omega_{\ell k}\underline{I}_k \tag{9}$$

and we obtain $\Omega_{\ell k}$, the array binary current-current interaction matrices, by solving the linear system (7) using a novel cyclic decomposition of the matrix Λ. The role of each array element and the presence of multiple reflections within subgroups of elements are clearly delineated by this method of solution. The array binary current-current interaction matrices $\Omega_{\ell k}$ defined in (9) can be interpreted as the total interaction matrix describing the interaction between elements ℓ and k in the array environment in terms of all Λ_{ij}. The $\Omega_{\ell k}$ are seen to be complicated but highly structured functions of Λ_{ij} that are amenable to physical interpretation. Matrix operators describing the role of the array environment can be easily identified, and from the recursive nature of the solution one discovers that the generalized current $\underline{I}_k^{(N)}$ at element k in an N element array is a linear combination of all the generalized currents $\underline{I}_\ell^{(N-1)}$ in an (N-1) element array. The complete solution to the N-element array problem leading to the radiation pattern can be constructed recursively and an inductive proof of the validity of the solution can be constructed. As an example, in Figure 1 we present and graphically interpret each of the five terms comprising $\boldsymbol{\Omega}_{21}^{(4)}$ in a 4-element array. Most of the features of the general N element solution are apparent in this example.

Finally, in terms of $\boldsymbol{\Lambda}_{ij}$ and $\boldsymbol{\Omega}_{ij}$, one can easily write the mutual impedance z_{ij} in (4). Thus,

$$z_{ij} = \delta_{ij} - \underset{-}{S}_{\alpha\beta}^{(i)\dagger}\Big(\sum_{k\neq\ell} Z_{\ell k}\, \boldsymbol{\Omega}_{kj}\Big)\Big[1 - \sum_{k\neq\ell} \Lambda_{\ell k}\, \boldsymbol{\Omega}_{kj}\Big]^{-1} \underset{-}{S}_{\beta\alpha}^{(j)} \qquad (10)$$

where δ_{ij} is the Kronecker delta.

References

[1] Kerns, D. M. Plane wave scattering-matrix theory of antennas and antenna-antenna interactions, NBS Monograph 162, U.S. Dept. of Commerce, National Bureau of Standards, Boulder, Colorado, 1981.

[2] Wasylkiwskyj, W. A network theory of coupling, radiation and scattering by antennas, Ph.D. thesis, Polytechnic Institute of Brooklyn, 1968, University Microfilms, Inc., Ann Arbor, Michigan.

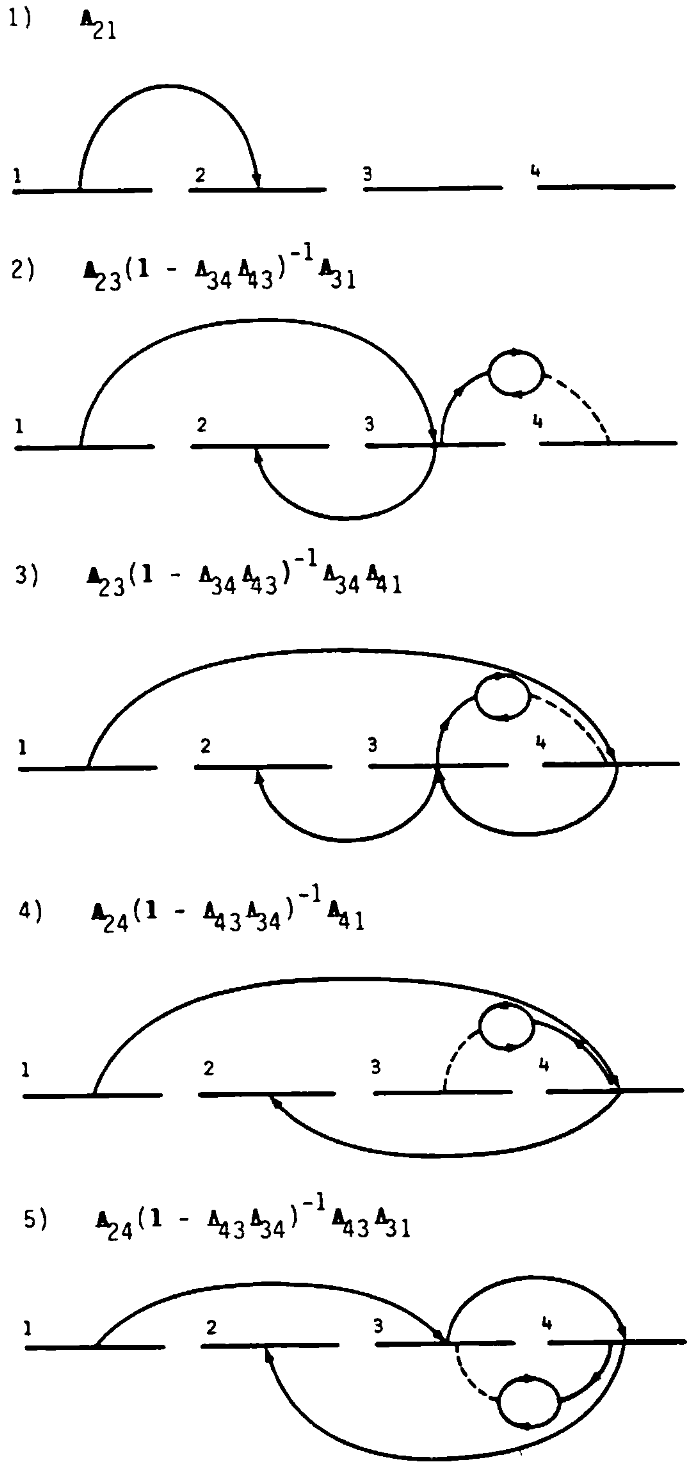

Figure 1. Possible signal paths and multiple reflections in a four element array.