A Near-Surface Interpolation Scheme Based on Radial Basis Function

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***Abstract-* The interpolation points adopted in the radial basis function (RBF) can be scattered. Based on the above fact, a near-surface interpolation scheme is proposed to combine RBFs for the scattering problems modeled with surface integral equations. The interpolation efficiencies of different RBFs with the proposed and Tartan grid schemes are compared to approximate the interactions between well-separated groups. It can be seen from the numerical results that the number of interpolation points is reduced significantly for all four RBFs, and the accuracy of the Gaussian RBF is better than the other three RBFs for different sizes of groups. The proposed scheme with GA RBF can be employed to build a fast solver.**

I. INTRODUCTION

The method of moments (MoM) using RWG basis functions [1-2] is widely used to solve 3-D electromagnetic scattering problems. Surface discretization is required in the MoM as opposed to volumetric discretization in the finite element method (FEM) and the finite difference time domain (FDTD) method. However, it is well known that the matrix given by the moment method is dense, which results in that high time cost is needed for the matrix-vector multiplications in an iterative solver. Hence, various fast algorithms [2] have been developed to accelerate the computing proceduce. The key step in these fast algorithms is to find an effective approximation of the interaction between two well-separated groups.

H-matrix [3] and H^2-matrix [4] techniques proposed recently can be regarded as general mathematical frameworks to combine various matrix approximations to accelerate an iterative solver. Lagrange polynomials interpolation can be employed to build a degenerate approximation of the interaction between two well-separated groups [5]. Similar works can be found in the multilevel Green's function interpolation method (MLGFIM) [6-9], a kernel independent approach, which has been successfully used to solve low-frequency [6] and full-wave electromagnetic problems [7]. Radial basis functions (RBFs) have been introduced to instead of the Lagrange polynomials to improve the computational performances of MLGFIM [7]. Furthermore, two different staggered interpolation point schemes [7-8] have been designed to reduce the number of interpolation points.

However, for the scattering problems modeled with surface integral equations, the unknowns are only distributed on the surface. The improvement of the computational efficiency is limited with the existing interpolation schemes, which are similar to the uniformly-spaced rectangular grids in adaptive integral method (AIM) [10]. To further reduce the number of interpolation points, a near-surface interpolation scheme is developed in this paper. The interpolation efficiencies of different RBFs with the proposed and Tartan grid schemes are compared. The shape parameter which is sensitive to the cube length and the number of interpolation points is also discussed.

II. THEORY

A. Radial Basis Function Interpolation

With the framework of H-matrix, a 3-D arbitrarily shaped object is recursively divided into smaller groups of different sizes. The well-separated groups are obtained according to their sizes and the distance of the two groups. Then the interaction between these groups can be approximated using interpolation technique

$$\tilde{\boldsymbol{G}}\left(\boldsymbol{r}_i,\boldsymbol{r}'_j\right)=\sum_{p=1}^{K}\sum_{q=1}^{K} w_p\left(\boldsymbol{r}_i\right) w_q\left(\boldsymbol{r}'_j\right) g\left(\boldsymbol{r}_{i,p},\boldsymbol{r}'_{j,q}\right) \tag{1}$$

where $g\left(\boldsymbol{r},\boldsymbol{r}'\right)=e^{-jkR}/4\pi R$ is the free space Green's function, $w_p\left(\boldsymbol{r}_i\right)$ and $w_q\left(\boldsymbol{r}'_j\right)$ are the *p*th and *q*th interpolation functions in the field group *i* and source group *j*, $\boldsymbol{r}_{i,p}$ and $\boldsymbol{r}'_{j,q}$ are the *p*th and *q*th interpolation points, respectively. Additionally, *K* is the number of interpolation points. Once the degenerate approximation is obtained, it can be integrated into H-matrix or H^2-matrix frameworks to establish a fast solver.

A function $f\left(\boldsymbol{r}\right)$ interpolated with RBFs can be expressed as follows:

$$f\left(\boldsymbol{r}\right)=\sum_{i=1}^{K}\beta_i\varphi_i\left(\boldsymbol{r}\right) \tag{2}$$

where $\{\varphi_i\}_{i=1}^{K}$ is a set of radial basis functions and $\{\beta_i\}_{i=1}^{K}$ are the corresponding coefficients. Several infinitely smooth RBFs are listed in Table I, *c* is the shape-controlling parameter of the functions.

Although it has been verified that RBFs interpolation can provide better interpolation accuracy than Lagrange polynomials interpolation, RBFs do not satisfy the Kronecker delta condition, i.e.,

$$\Phi_j\left(\boldsymbol{r}_i\right)=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases} \tag{3}$$

TABLE I
INFINITE SMOOTH RBFs

Type of RBFs	$\varphi_i(\boldsymbol{r})$
Gaussian (GA)	$e^{-c\lvert\boldsymbol{r}-\boldsymbol{r}_i\rvert^2}$
Inverse quadratic (IQ)	$1\Big/\left(\lvert\boldsymbol{r}-\boldsymbol{r}_i\rvert^2+c^2\right)$
Inverse multiquadric (IMQ)	$1\Big/\sqrt{\lvert\boldsymbol{r}-\boldsymbol{r}_i\rvert^2+c^2}$
Multiquadric (MQ)	$\sqrt{\lvert\boldsymbol{r}-\boldsymbol{r}_i\rvert^2+c^2}$

which is important for the construction of high quality interpolation, and makes it easier to get the coefficients in (2). In order to obtain the orthogonal RBFs, a normalized orthonormalization can be utilized to get a set of auxiliary functions $\{\psi_i\}_{i=1}^K$, which is a linear combination of the RBFs and satisfies Eq. (3).

$$\psi_i(\boldsymbol{r})=\sum_{j=1}^{K}\gamma_{i,j}\varphi_j(\boldsymbol{r}) \tag{4}$$

Substituting (4)into (3), a matrix equation can be obtained,

$$\bar{\bar{\gamma}}\cdot\bar{\bar{\varphi}}=\begin{bmatrix}\gamma_{1,1} & \cdots & \gamma_{1,K}\\ \vdots & \ddots & \vdots\\ \gamma_{K,1} & \cdots & \gamma_{K,K}\end{bmatrix}\begin{bmatrix}\varphi_1(\boldsymbol{r}_1) & \cdots & \varphi_1(\boldsymbol{r}_K)\\ \vdots & \ddots & \vdots\\ \varphi_K(\boldsymbol{r}_1) & \cdots & \varphi_K(\boldsymbol{r}_K)\end{bmatrix}=\bar{\bar{I}} \tag{5}$$

where $\bar{\bar{I}}$ is a *K-by-K* unitary matrix. The singular valued decomposition (SVD) algorithm [11] is adopted to get the coefficients $\gamma_{i,j}$ because the matrix $\bar{\bar{\varphi}}$ consisted of RBFs is ill conditioned for a large matrix size *K*. Then the normalized orthogonal auxiliary functions $\psi_i(\boldsymbol{r})$ are obtained and employed to interpolate the function $f(\boldsymbol{r})$.

$$f(\boldsymbol{r})=\sum_{i=1}^{K}f(\boldsymbol{r}_i)\psi_i(\boldsymbol{r}) \tag{6}$$

In this way, the functions $w_p(\boldsymbol{r}_i)$ and $w_q(\boldsymbol{r}_j')$ in (1) can be expressed in the interpolation form [7-8].

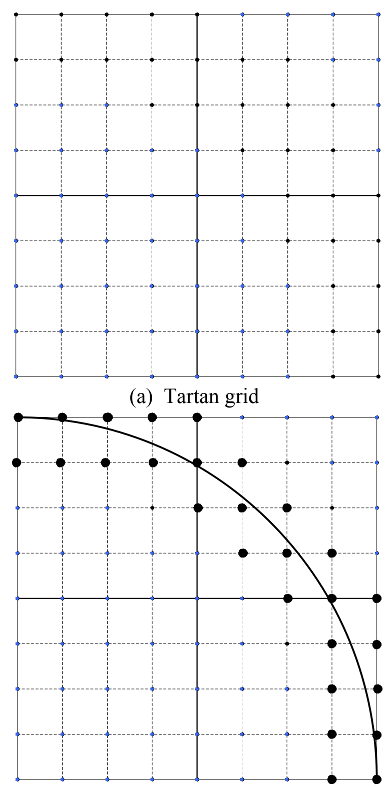
B. Near-Surface Interpolation Scheme

From the interpolation theory, it is known that the function value of a point is a linear combination of interpolation basis functions, and such a value can mainly be determined by contributions of the nearby interpolation points.

Fig. 1(a) shows 2-dimensional Tartan grid. The number of the interpolation points marked with small dots within Tartan grid is determined by the size of the group region. The two different staggered Tartan grids proposed in [7-8] involve all the interpolation points in the region. However, for solving surface integral equations, only part of the points is near the surface, which can be seen from Fig. 1(b). The surface is represented by the curve. Most of the interpolation points within Tartan grid are far away from the surface. It is found that good interpolation accuracy can still be achieved without the interpolation points far away from the surface. Hence, a near-surface interpolation scheme is proposed, which is illustrated with bigger black dots in Fig. 1(b). The required interpolation points for interpolation can be limited with the condition

$$\lvert\boldsymbol{r}_i-\boldsymbol{r}_s\rvert\le\alpha d \tag{7}$$

where $\boldsymbol{r}_i$ and $\boldsymbol{r}_s$ are the points from Tartan grid and the surface, d is the distance of two adjacent interpolation points, α is a factor used to control the bounds of interpolation points. With the increase of the factor α, the proposed scheme behaves more similar to Tartan grid. A small α may give low interpolation accuracy. It is recommended that the factor α is set to 1. 02.



(a) Tartan grid

(b) the proposed scheme

Figure 1. 2-dimensional interpolation point schemes

III. NUMERICAL RESULTS AND DISCUSSION

The accuracy and efficiency of the near-surface interpolation scheme are verified with the test examples in this section. The results of different RBFs with Tartan grid are given as a reference, and then the interpolation relative errors of different RBFs with the proposed scheme are calculated.

In the first example, the interpolation efficiencies of different RBFs with Tartan grid are compared for groups of different electrical sizes. The results are listed in Table II. The interpolation error threshold is set to 0.02. The interpolation relative error is calculated with the following formula,

$$\varepsilon \approx \frac{\left\| \boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}') - \tilde{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}') \right\|_F}{\left\| \boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}') \right\|_F} \tag{8}$$

where $\tilde{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')$ is the interpolated approximation of $\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}')$, $\|\bullet\|_F$ denotes the Frobenius norm.

From Table II, it is found that the interpolation with the Gaussian (GA) RBF obtains better interpolation accuracy than the other three kinds of RBFs, which is consistent with the results in [7-8].

The efficiency comparisons of different RBFs with the near-surface interpolation scheme are listed in Table III. In this example, spheres with diameters of different wavelengths are divided uniformly into 8 groups of different sizes. It can also be found that the interpolation with the GA RBF obtains the best performance, and less interpolation point numbers are needed for all four RBFs to satisfy the error threshold.

It can be seen from Table II and III that the numbers of interpolation points are reduced drastically for all RBFs with the proposed scheme, and the increase of the interpolation point number of the proposed scheme is much slower than the Tartan grid with the increase of group sizes, which can be seen explicitly in Fig. 2. When the group lengths are larger than 2.5λ, it is more difficult for Tartan grid to obtain the pseudo inverse using the singular valued decomposition (SVD) algorithm than the proposed scheme. Hence, the near-surface interpolation scheme with the GA RBF is more suitable for interpolation of large groups.

Fig. 3 shows the interpolation error of the GA RBF versus the shape parameter c of groups with length of 2 wavelengths. A sphere with a diameter of 4 wavelengths is demonstrated, and similar results can be observed for other electrical sizes. The interpolation point numbers of the proposed scheme are about 251, which vary slightly for different groups. Although the surface distributions in 8 groups are different, the same shape parameter can be adopted to satisfy the error threshold, which is very important for the proposed scheme to be employed to build a fast solver.

TABLE II
EFFICIENCY COMPARISONS OF DIFFERENT RBFS

Type of basis function	Length of groups (λ)	Number of interpolation points	Optimized shape parameter	Interpolation relative error
GA RBF	0.5	64	2.80	0.00426
	1.0	125	2.40	0.0140
	1.5	343	1.50	0.00434
	2.0	512	1.40	0.00827
	2.5	729	1.30	0.0132
IQ RBF	0.5	64	1.00	0.00618
	1.0	216	1.60	0.00362
	1.5	343	1.70	0.00841
	2.0	512	1.90	0.0154
	2.5	1000	2.60	0.00837
IMQ RBF	0.5	64	0.90	0.00643
	1.0	216	1.50	0.00379
	1.5	343	1.70	0.00862
	2.0	512	1.80	0.0160
	2.5	1000	2.50	0.00870
MQ RBF	0.5	64	0.70	0.00725
	1.0	216	1.30	0.00422
	1.5	343	1.50	0.00961
	2.0	512	1.60	0.0175
	2.5	1000	2.20	0.00969

TABLE III
EFFICIENCY COMPARISONS OF DIFFERENT RBFS WITH NEAR-SURFACE INTERPOLATION SCHEME

Type of basis function	Length of groups (λ)	Number of interpolation points	Optimized shape parameter	Interpolation relative error
GA RBF	0.5	43	2.90	0.00468
	1.0	73	2.30	0.0116
	1.5	145	1.40	0.0131
	2.0	251	1.20	0.0188
	2.5	371	1.20	0.0167
	3.0	515	1.10	0.0144
	4.0	885	1.00	0.0200
IQ RBF	0.5	43	1.20	0.0119
	1.0	106	1.60	0.0124
	1.5	184	2.30	0.0133
	2.0	304	2.50	0.0190
	2.5	427	2.40	0.0186
	3.0	609	2.30	0.0179
	4.0	992	2.20	0.0197
IMQ RBF	0.5	43	1.00	0.0127
	1.0	106	1.50	0.0130
	1.5	184	2.30	0.0134
	2.0	304	2.30	0.0200
	2.5	427	2.10	0.0176
	3.0	609	2.00	0.0178
	4.0	992	2.00	0.0203
MQ RBF	0.5	43	0.90	0.0146
	1.0	106	1.40	0.0145
	1.5	184	2.10	0.0149
	2.0	304	2.10	0.0221
	2.5	427	1.90	0.0185
	3.0	609	1.80	0.0191
	4.0	992	1.60	0.0220

IV. CONCLUSION

In this paper, a near-surface interpolation scheme is presented based on the radial basis functions, which can reduce the interpolation points significantly. The interpolation efficiencies of the proposed and Tartan grid schemes are compared with different RBFs. Numerical results show that the performance of the Gaussian RBF is better than the other three RBFs when they are used to approximate the free space Green's function. The presented interpolation technique can be employed to construct a degenerate kernel for electromagnetic surface integral equations, for establishment of a fast solving algorithm.

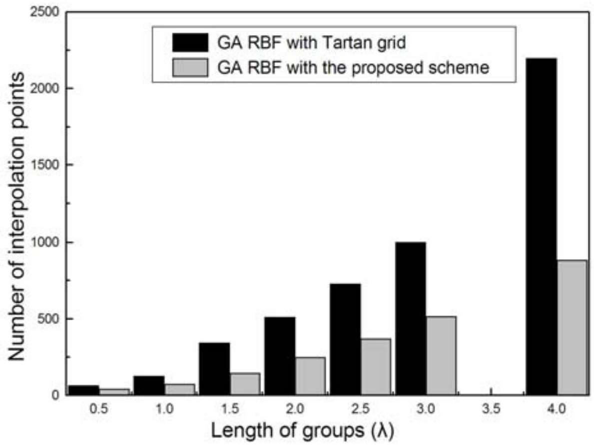


Figure 2. Variation of the number of interpolation points with groups of different wavelengths

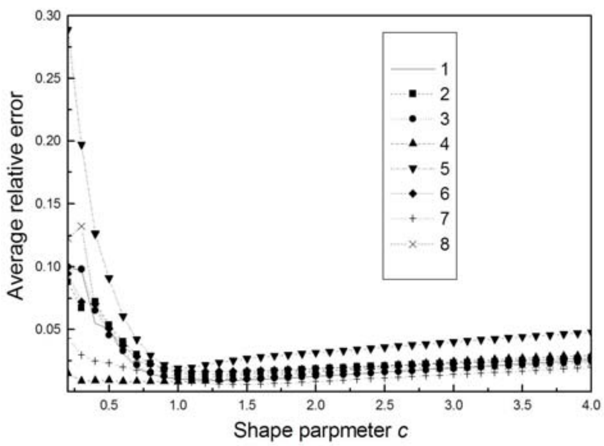


Figure 3. Average relative error versus the shape parameter *c*

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