IEICE Proceeding Series

A Theory on Noise-Induced Synchronization of Chaotic Oscillators

Wataru Kurebayashi, Kantaro Fujiwara, Hiroya Nakao, Tohru Ikeguchi

Vol. 1 pp. 344-347
Publication Date: 2014/03/17
Online ISSN: 2188-5079

2012 International Symposium on Nonlinear Theory and its Applications
NOLTA2012, Palma, Majorca, Spain, October 22-26, 2012

A Theory on Noise-Induced Synchronization of Chaotic Oscillators

Wataru Kurebayashi†, Kantaro Fujiwara†, Hiroya Nakao†† and Tohru Ikeguchi†,‡

†Graduate School of Science and Engineering, Saitama University
Shimo-Ohkubo 255, Sakura-ku, Saitama city, Saitama, 338-8570
††Department of Mechanical and Environmental Informatics, Tokyo Institute of Technology
O-okayama 2-12, Meguro-ku, Tokyo, 152-8552
‡Saitama University Brain Science Institute
Shimo-Ohkubo 255, Sakura-ku, Saitama city, Saitama, 338-8570
Email: kure@nls.ics.saitama-u.ac.jp

Abstract—Phase description is an essential tool for analytically investigating synchronization phenomena of limit cycle oscillators. In this paper, introducing a new type of phase description, we discussed the noise-induced phase synchronization of strongly fluctuating oscillators such as chaotic oscillators. We derived a probability density function of phase differences between oscillators, which enables us to explore statistical properties of the synchronization phenomena.

1. Introduction

Synchronization is a ubiquitous phenomenon in the real world. We can observe synchronization phenomena of fireflies [1], frogs [2], slime mold [3] and neurons [4]. To analytically investigate these synchronization phenomena, the phase description is very useful [5]. The phase description enables us to reduce higher-dimensional dynamics to a simple equation with one degree of freedom, which is called a phase equation.

However, we cannot directly apply the conventional phase description to strongly fluctuating oscillators such as chaotic oscillators. Although many previous works [6, 7, 8] numerically explore the phase synchronization of chaotic oscillators, analytical frameworks have not been established yet. Recently, several works have tried to apply the phase description to chaotic oscillators [9, 10, 11]. We must note that the chaotic oscillators stochastically response to external forces in the sense that its response is not only determined by its phase. To discuss the synchronization of chaotic oscillators, it is important to take this stochastic response into accounts. Previous works cannot effectively characterize this stochasticity.

In this paper, we introduce a new type of phase description, which effectively describes the dynamics of strongly fluctuating oscillators. We derived an effective phase equation that reproduces the long-time dynamics of such oscillators. In our theory, we use a continuous spectrum to characterize the stochastcity of the response of chaotic oscillators, which successfully reproduces statistical properties of chaotic oscillators.

Applying our theory to chaotic oscillators, we discuss the noise-induced phase synchronization of chaotic oscillators. We derived the probability density function of phase differences between oscillators, which enables us to explore statistical properties of various synchronization phenomena.

2. Phase Description

2.1. Introducing a phase equation

We consider a chaotic oscillator described by

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}) + \sigma \boldsymbol{G}(\boldsymbol{X}) p(t), \qquad (1)$$

where $\boldsymbol{X} \in \mathbb{R}^n$ is an n-dimensional state variable, $\boldsymbol{F}(\boldsymbol{X}) \in \mathbb{R}^n$ is an unperturbed vector field, $p(t) \in \mathbb{R}$ is an external force, σ is a parameter to control the intensity of the external force, and $\boldsymbol{G}(\boldsymbol{X}) \in \mathbb{R}^n$ represents the coupling of the oscillators to the external force.

Here, we introduce a phase $\phi(t) = \phi(\boldsymbol{X}(t)) \in [0, 2\pi)$ as a function of the state variable $\boldsymbol{X}(t)$. Using the chain rule, we can obtain an equation describing the dynamics of $\phi(t)$ as

$$\dot{\phi} = \mathrm{grad}_{\boldsymbol{X}} \phi(\boldsymbol{X}(t)) \cdot \dot{\boldsymbol{X}} = \omega(t) + \sigma \zeta(t) p(t), \qquad (2)$$

where

$$\omega(t) = \mathrm{grad}_{\boldsymbol{X}} \phi(\boldsymbol{X}(t)) \cdot \boldsymbol{F}(\boldsymbol{X}(t)), \qquad (3)$$
$$\zeta(t) = \mathrm{grad}_{\boldsymbol{X}} \phi(\boldsymbol{X}(t)) \cdot \boldsymbol{G}(\boldsymbol{X}(t)). \qquad (4)$$

Although we have many options to define the function $\phi(\boldsymbol{X}(t))$, we employ the definition proposed by Schwabedal *et al.* [11]. In this definition, the phase $\phi(\boldsymbol{X}(t))$ is defined so that an integrated square error $\int |2\pi t/T - \phi(\boldsymbol{X}(t))|^2 dt$ is minimized, where T is a mean period of the oscillator.

In the conventional phase description, under the assumption that the system of Eq. (1) has a limit cycle solution $\boldsymbol{\chi}(t)$ and a perturbation is very weak ($\sigma \ll 1$), we can approximately rewrite Eq. (2) to the following equation:

$$\dot{\phi} = \omega + \sigma z(\phi)p(t), \tag{5}$$

where ω is a constant called a natural frequency, and $z(\phi)$ is a function of $\phi(t)$ called a phase sensitivity and is defined as $z(\phi) = \mathrm{grad}_{\boldsymbol{X}}\phi(\boldsymbol{\chi}(\phi)) \cdot \boldsymbol{G}(\boldsymbol{\chi}(\phi))$.

However, we cannot apply the conventional phase description as demonstrated in Eq. (5) to chaotic oscillators, because $\omega(t)$ is not constant and $\zeta(t)$ is not an explicit function of $\phi(t)$ in case of the chaotic oscillators. We need a different type of phase description that effectively describes the stochasticity of $\omega(t)$ and $\zeta(t)$, namely, the effective phase description. In the following sections, we regard $\omega(t)$ and $\zeta(t)$ as stochastic processes, and derive a closed-form equation of the phase $\phi(t)$.

2.2. Natural frequency

We define $\hat{\omega}$ as a temporal average of $\omega(t)$ as follows:

$$\hat{\omega} = \langle \omega(t) \rangle_t, \tag{6}$$

where $\langle \cdot \rangle_t$ represents a temporal average defined as $\langle \cdot \rangle_t = \lim_{\tau \to \infty} (2\tau)^{-1} \int_{-\tau}^{+\tau} \cdot \, dt$. In this paper, we call $\hat{\omega}$ a natural frequency. Here, we further define a diffusion coefficient D_ω as follows:

$$D_\omega = \int_{-\infty}^{+\infty} \langle [\omega(t) - \hat{\omega}][\omega(t-s) - \hat{\omega}] \rangle_t ds. \tag{7}$$

which is called a natural diffusion coefficient in this paper. In the long-time limit ($t_1 - t_0 \gg 1$), we can apply the diffusion approximation to $\omega(t)$ as follows:

$$\mathrm{E}\left[\left(\int_{t_0}^{t_1} [\omega(t) - \hat{\omega}] dt \right)^2 \right] \simeq D_\omega (t_1 - t_0), \tag{8}$$

where $\mathrm{E}[\cdot]$ is the expectation over realizations of the stochastic process $\omega(t)$.

Thus, we can approximate $\omega(t)$ as

$$\omega(t) \simeq \hat{\omega} + \eta_\omega(t), \tag{9}$$

where $\eta_\omega(t)$ is Gaussian white noise that satisfies $\langle \eta_\omega(t) \rangle_t = 0$ and $\langle \eta_\omega(t) \eta_\omega(t-\tau) \rangle_t = D_\omega \delta(\tau)$.

2.3. Phase sensitivity

We define a phase sensitivity $\hat{\zeta}(\phi)$ as follows:

$$\hat{\zeta}(\psi) = \mathrm{E}[\zeta(t) | \phi(t) = \psi], \tag{10}$$

where $\mathrm{E}[A|B]$ is the expectation of A under the condition of B. We can estimate the phase sensitivity $\hat{\zeta}(\phi)$ by experiments in the same way as a phase sensitivity $z(\phi)$ in the conventional phase description. Using $\hat{\zeta}(\phi)$, we can rewrite a phase sensitivity $\zeta(t)$ as

$$\zeta(t) = \hat{\zeta}(\phi) + \eta_\zeta(t), \tag{11}$$

where $\eta_\zeta(t)$ is a stochastic process defined as $\eta_\zeta(t) = \zeta(t) - \hat{\zeta}(\phi)$. By substituting Eq. (11) into $\zeta(t)p(t)$, we can obtain

$$\zeta(t)p(t) = \hat{\zeta}(\phi)p(t) + \eta_\zeta(t)p(t). \tag{12}$$

We assume that the stochastic processes $\eta_\zeta(t)$ is approximately independent from the external force $p(t)$. Under this assumption, it holds that $\langle \eta_\zeta(t)p(t) \rangle = \langle \eta_\zeta(t) \rangle \langle p(t) \rangle = 0$, because $\langle \eta_\zeta(t) \rangle = 0$.

Here, we define a diffusion coefficient $D_{\zeta p}$ as follows:

$$\begin{aligned} D_{\zeta p} &= \int_{-\infty}^{+\infty} \big\langle [\eta_\zeta(s)p(s) - \langle \eta_\zeta(t)p(t) \rangle_t] \\ &\quad \times [\eta_\zeta(s-u)p(s-u) - \langle \eta_\zeta(t)p(t) \rangle_t] \big\rangle_s du \\ &= \int_{-\infty}^{+\infty} \langle \eta_\zeta(s)\eta_\zeta(s-u) \rangle_s \\ &\quad \times \langle p(s)p(s-u) \rangle_s du. \end{aligned} \tag{13}$$

We also define correlation functions $C_\zeta(u)$ and $C_p(u)$ as $C_\zeta(u) = \langle \eta_\zeta(s)\eta_\zeta(s-u) \rangle_s$ and $C_p(u) = \langle p(s)p(s-u) \rangle_s$. Then, using the correlation functions $C_\zeta(u)$ and $C_p(u)$, we can rewrite Eq. (13) as

$$D_{\zeta p} = \int_{-\infty}^{+\infty} C_\zeta(t) C_p(t) dt. \tag{14}$$

In the long-time limit ($t_1 - t_0 \gg 1$), we can apply the diffusion approximation to $\zeta(t)p(t)$ as follows:

$$\mathrm{E}\left[\left(\int_{t_0}^{t_1} \eta_\zeta(t)p(t) dt \right)^2 \right] \simeq D_{\zeta p}(t_1 - t_0), \tag{15}$$

where $\mathrm{E}[\cdot]$ is the expectation over realizations of the stochastic process $\eta_\zeta(t)$.

Thus, we can approximate $\zeta(t)p(t)$ as follows:

$$\zeta(t)p(t) \simeq \hat{\zeta}(\phi)p(t) + \eta_{\zeta p}(t), \tag{16}$$

where $\eta_{\zeta p}(t)$ is Gaussian white noise that satisfies $\langle \eta_{\zeta p}(t) \rangle_t = 0$ and $\langle \eta_{\zeta p}(t) \eta_{\zeta p}(t-\tau) \rangle_t = D_{\zeta p}\delta(\tau)$. Here, we define the power spectra $P_\zeta(\Omega)$ and $P_p(\Omega)$ as $P_\zeta(\Omega) = \int_{-\infty}^{+\infty} e^{-i\Omega\tau} C_\zeta(\tau) d\tau$ and $P_p(\Omega) = \int_{-\infty}^{+\infty} e^{-i\Omega\tau} C_p(\tau) d\tau$. Using the power spectra $P_\zeta(\Omega)$ and $P_p(\Omega)$, we can rewrite Eq. (14) to Eq. (17).

$$D_{\zeta p} = \int_{-\infty}^{+\infty} P_\zeta(\Omega) P_p(\Omega) d\Omega. \tag{17}$$

The Fourier representation as in Eq. (17) is useful, because $P_\zeta(\Omega)$ can be easily estimated as we will discuss in section 2.5. In this paper, we call $P_\zeta(\Omega)$ a phase sensitivity spectrum.

2.4. Effective phase equation

Substituting Eqs. (9) and (16) into Eq. (2), we can obtain the following phase equation for strongly fluctuating oscillators.

$$\dot{\phi} = \hat{\omega} + \sigma\hat{\zeta}(\phi)p(t) + \eta(t), \quad (18)$$

where $\eta(t)$ is Gaussian white noise defined as $\eta(t) = \eta_\omega(t) + \sigma\eta_{\zeta p}(t)$ and satisfies $\langle\eta(t)\rangle_t = 0$ and $\langle\eta(t)\eta(t-\tau)\rangle_t = (D_\omega + \sigma^2 D_{\zeta p})\delta(\tau) = (D_\omega + \sigma^2\int_{-\infty}^{\infty} P_\zeta(\Omega)P_p(\Omega)d\Omega)\delta(\tau)$. Eq. (18) is a closed-form equation of the phase ϕ. Thus, it is easy to analyze the dynamics and statistical properties of Eq. (18).

In this paper, we call Eq. (18) an effective phase equation. Using Eq. (18), we can characterize the dynamics of a strongly fluctuating oscillator only by a natural frequency $\hat{\omega}$, a phase sensitivity $\hat{\zeta}(\phi)$, a natural diffusion coefficient D_ω and a phase sensitivity spectrum $P_\zeta(\Omega)$. This simple equation would be a strong tool to investigate synchronization phenomena of chaotic oscillators.

2.5. Estimation of a phase sensitivity spectrum

We define a diffusion coefficient of the phase ϕ, D_ϕ, as follows:

$$D_\phi = \int_{-\infty}^{+\infty} \Big\langle[\phi(s) - \langle\phi(t)\rangle_t] \times[\phi(s-u) - \langle\phi(t)\rangle_t]\Big\rangle_s du. \quad (19)$$

We further define the Fourier expansion of a phase sensitivity $\hat{\zeta}(\phi)$ as follows:

$$\hat{\zeta}(\phi) = \sum_{l=-\infty}^{+\infty} \hat{\zeta}_l e^{il\phi}, \quad (20)$$

where $\hat{\zeta}_l$ $(l = -\infty, \ldots, \infty)$ are Fourier coefficients of $\hat{\zeta}(\phi)$. If we assume that $p(t)$ is a colored noise that has a power spectrum $P_p(\Omega)$, as discussed in Ref. [12], we can obtain the diffusion coefficient D_ϕ as follows:

$$D_\phi = D_\omega + \sigma^2\int_{-\infty}^{+\infty} P_\zeta(\Omega)P_p(\Omega)d\Omega + \sigma^2\sum_{l=-\infty}^{+\infty}|\hat{\zeta}_l|^2 P_p(l\hat{\omega}). \quad (21)$$

Here, we assume that $P_p(t) = P_{\rm ex}(\Omega;\gamma,\omega_0)$, where $P_{\rm ex}(\Omega;\gamma,\omega_0) = (\gamma^2/2)\{1/[\gamma^2 + (\Omega+\omega_0)^2] + 1/[\gamma^2 + (\Omega-\omega_0)^2]\}$ and γ and ω_0 are parameters. This type of colored noise can be easily generated [12]. If γ is sufficiently small $(\gamma \to 0)$, we have $P_{\rm ex}(\Omega;\gamma,\omega_0) \to (\pi\gamma/2)[\delta(\Omega+\omega_0)+\delta(\Omega-\omega_0)]$. Thus, if we can measure D_ϕ of a chaotic oscillator subject to colored noise $p(t)$ by experiments, we can estimate a phase sensitivity spectrum $P_\zeta(\Omega)$ only from D_ϕ, D_ω and $\hat{\zeta}(\phi)$ as follows:

$$P_\zeta(\omega_0) \simeq \frac{1}{\sigma^2\pi\gamma}\Bigg[D_\phi - D_\omega - \sigma^2\sum_{l=-\infty}^{+\infty}|\hat{\zeta}_l|^2 P_{\rm ex}(l\hat{\omega};\gamma,\omega_0)\Bigg], \quad (22)$$

where the diffusion coefficients D_ϕ and D_ω can be measured by experiments [12].

3. Noise-Induced Phase Synchronization

To demonstrate how our theory works well, we applied the theory to the noise-induced synchronization of chaotic oscillators. We consider an ensemble of N identical chaotic oscillators described by

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}(\boldsymbol{X}_i) + \boldsymbol{G}(\boldsymbol{X}_i)[p(t) + q_i(t)], \quad (23)$$

for $i = 1, \ldots, N$, where $\boldsymbol{X}_i$ is the state of the i-th oscillator, $p(t)$ is common noise, and $q_i(t)$ is independent noise. For simplicity, we assume that $\langle p(t)\rangle_t = \langle q_i(t)\rangle_t = 0$, $\langle p(t)q_i(t-\tau)\rangle_t = 0$, $\langle q_i(t)q_j(t-\tau)\rangle_t = 0$ $(i \neq j)$, and $q_i(t)$ $(i = 1, \ldots, N)$ have the same statistical property characterized by a power spectrum $P_q(\Omega)$. By the effective phase description, we can reduce Eq. (23) to the following effective phase equation:

$$\dot{\phi}_i = \hat{\omega} + \hat{\zeta}(\phi)[p(t) + q_i(t)] + \eta(t), \quad (24)$$

for $i = 1, \ldots, N$, where ϕ_i is the phase of the i-th oscillator.

Here, we focus on the two-body problem of ϕ_1 and ϕ_2, and define a phase difference θ as $\theta = \phi_1 - \phi_2$. By discussing the dynamics of the phase difference θ, we can analytically explore the statistical property of synchronization phenomena without loss of generality. As demonstrated in Ref. [13], we can obtain the probability density function of the phase difference θ, $f(\theta)$, as follows:

$$f(\theta) = \frac{c}{g(0) - g(\theta) + h(0) + D_\eta}, \quad (25)$$

where c is a normalization constant, $g(\theta)$ and $h(\theta)$ are correlation functions defined as

$$g(\theta) = \sum_{l=-\infty}^{+\infty}|\hat{\zeta}_l|^2 P_p(l\hat{\omega})e^{il\hat{\omega}}, \quad (26)$$

$$h(\theta) = \sum_{l=-\infty}^{+\infty}|\hat{\zeta}_l|^2 P_q(l\hat{\omega})e^{il\hat{\omega}}, \quad (27)$$

and D_η is a diffusion coefficient defined as

$$D_\eta = D_\omega + \int_{-\infty}^{+\infty} P_\zeta(\Omega)P_p(\Omega)d\Omega. \quad (28)$$

Let us note that our theory is valid under the condition that a perturbation is sufficiently weak ($\sigma \ll 1$) so that $\eta_\zeta(t)$ of each oscillator can be assumed to be independent.

4. Summary and Discussions

In this paper, we introduced a new type of phase description. Using our theory, we can obtain a closed-form equation of the phase, which we call an effective phase equation. An effective phase equation describes the long-time dynamics of a strongly fluctuating oscillator such as a chaotic oscillator. The effective phase equation is characterized only by two constants and two functions, that is, a natural frequency $\hat{\omega}$, a natural diffusion coefficient D_ω, a phase sensitivity $\hat{\zeta}(\phi)$ and a phase sensitivity spectrum $P_\zeta(\Omega)$. Thus, we can possibly apply this equation to various phenomena in the real world.

Using the effective phase description, we discussed the noise-induced phase synchronization of strongly fluctuating oscillators. We derived the probability density function of phase differences between oscillators, which enables us to explore statistical properties of various synchronization phenomena. Recently, the noise-induced synchronization of chaotic oscillators is applied to engineering problems such as secure key distribution [14]. Thus, our theory would be useful for a wide range of purposes from mathematical modelings to engineering applications.

In the real world, various biological and chemical oscillations are described as limit cycle oscillators. However, we cannot directly apply the conventional phase description to these oscillators, because they are subject to sufficiently large noise, which is inevitable in real-world environments. Our theory is a possible analytical tool to explore such stochastic oscillation phenomena. In particular, one of the possible future work is to apply the theory to neuroscience. To achieve this goal, we have to develop an effective estimation method of a phase sensitivity spectrum $\hat{\zeta}(\phi)$.

Acknowledgments

The authors would like to thank S. Ogawa and AGS Corporation for their kind encouragement on this research project. The authors are also grateful to K. Yoshimura for fruitful discussions.

References

[1] S. H. Strogatz and I. Stewart. Coupled oscillators and biological synchronization. *Sci. Am.*, 269(6):102–109, 1993.

[2] I. Aihara. Modeling synchronized calling behavior of Japanese tree frogs. *Phys. Rev. E*, 80:011918, 2009.

[3] A. Takamatsu, T. Fujii, and I. Endo. Time delay effect in a living coupled oscillator system with the plasmodium of *Physarum polycephalum.* *Phys. Rev. Lett.*, 85:2026–2029, Aug 2000.

[4] E. Brown, J. Moehlis, and P. Holmes. On the phase reduction and response dynamics of neural oscillator populations. *Neural Comput.*, 16(4):673–715, 2004.

[5] Y. Kuramoto. *Chemical oscillations, waves, and turbulence.* Springer-Verlag, Berlin, 1984.

[6] M. Rosenblum, A. Pikovsky, and J. Kurths. Phase synchronization of chaotic oscillators. *Phys. Rev. Lett.*, 76(11):1804–1807, 1996.

[7] C. Zhou, J. Kurths, I. Z. Kiss, and J. L. Hudson. Noise-enhanced phase synchronization of chaotic oscillators. *Phys. Rev. Lett.*, 89(1):014101, Jun 2002.

[8] Y. Wang, Y. C. Lai, and Z. Zheng. Onset of colored-noise-induced synchronization in chaotic systems. *Phys. Rev. E*, 79(5):056210, 2009.

[9] T. Imai, H. Suetani, and T. Aoyagi. A phase reduction approach for synchronization of chaotic systems. *IEICE Tech. Rep.*, 109(269):97–102, 2009.

[10] S. Hata, K. Arai, and H. Nakao. Stochastic phase description of fluctuating rhythmic elements. *IEICE Tech. Rep.*, 110(82):135–139, 2010.

[11] J. T. C. Schwabedal, A. Pikovsky, B. Kralemann, and M. Rosenblum. Optimal phase description of chaotic oscillators. *Phys. Rev. E*, 85:026216, 2012.

[12] H. Nakao, J.-N. Teramae, D. S. Goldobin, and Y. Kuramoto. Effective long-time phase dynamics of limit-cycle oscillators driven by weak colored noise. *Chaos*, 20(3):3126, 2010.

[13] W. Kurebayashi, K. Fujiwara, and T. Ikeguchi. Colored noise induces synchronization of limit cycle oscillators. *Europhys. Lett.*, 97:50009, 2012.

[14] K. Yoshimura, J. Muramatsu, P. Davis, T. Harayama, H. Okumura, S. Morikatsu, H. Aida, and A. Uchida. Secure key distribution using correlated randomness in lasers driven by common random light. *Phys. Rev. Lett.*, 108:070602, 2012.