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Keiji Okumura, Kazuyuki Aihara

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A solvable model for noise-induced synchronization in ensembles of coupled excitable oscillators

Keiji Okumura[†‡] and Kazuyuki Aihara[‡]

†FIRST, Aihara Innovative Mathematical Modelling Project, Japan Science and Technology Agency
4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, Japan
‡Institute of Industrial Science, The University of Tokyo
4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, Japan
Email: kokumura@sat.t.u-tokyo.ac.jp

Abstract—A mean-field model for coupled excitable oscillators is introduced to analyze noise-induced synchronization. A nonlinear Fokker-Planck equation approach allows us to study the effects of noise on the system only by dealing with deterministic nonlinear dynamics. Taking the thermodynamic limit, we derive the time evolution equations of the order parameters for the system without any approximations. A bifurcation diagram for the order parameters is shown compared to collective behavior of the oscillators obtained by direct simulation of the set of the Langevin equations with changes in noise intensity.

1. Introduction

Effects of noise on synchronization phenomena in oscillatory systems have attracted much attention in many fields of natural sciences in recent years. On one hand, one can easily suppose that the presence of noise might deteriorate the degree of synchronization of oscillatory systems. On the other hand, the counterintuitive phenomena of noise induced synchronization are intensively studied in many types of oscillatory systems [1–3]. Synchronization phenomena induced by independent white noise in coupled excitable oscillators including active rotator models are investigated both analytically [4–10] and numerically [11].

To understand how noise exerts its influence on the structure of synchronization will be of great importance for the purpose of nonlinear dynamical controls involving changes in synchrony of oscillatory systems. So far as we know, there are very few papers that concern analytical studies of the relationships between noise effects and synchronization in the case of coupled excitable oscillators without any approximations.

For the definition of synchronization in coupled excitable oscillator systems under the influence of noise, let us consider relationships between behavior of statistical quantities and dynamical variables in oscillatory systems. In a deterministic system of the coupled excitable oscillators, the synchronization phenomena are well-defined by the condition that all of the dynamical variables of the system take periodic motion with a common time period. However, in a stochastic system of a finite number of coupled excitable oscillators which is subjected to independent noise, each oscillator behaves randomly due to the presence of noise. The corresponding Fokker-Planck equation [12] involving all of the dynamical variables which describe the system of many body oscillators is linear. The probability density of the system as its solution, in general, exhibits ergodic property to settle into equilibrium (*i.e.* fixed point type) probability density for sufficiently large times. Order parameters of the system do not oscillate even if individual oscillators periodically behave in the deterministic case. Therefore it seems to be difficult to appropriately define the synchronization phenomena in this situation.

To consider the synchronization phenomena in a system of coupled excitable oscillators with noise by overcoming the problem mentioned above, let us introduce the concept of taking the thermodynamic limit with mean-field coupling. Taking advantage of these issues, it is useful to employ the nonlinear Fokker-Planck equation (NFPE) approach [13–16]. In a mean-field coupling model where all of the oscillators are coupled by each other, the law of large numbers can be applied. As a consequence, the Fokker-Planck operator itself includes empirical probability density of the system and the Fokker-Planck equation becomes nonlinear. Now that the system loses ergodicity, it can exhibit a rich variety of bifurcations corresponding to nonequilibrium phase transitions. The empirical probability density of a coupled oscillatory system might be multimodal with weak external noise. However, in some solvable models with the validity of an H theorem for an NFPE [10, 17, 18] or under the assumption of Gaussian approximations [8], it takes unimodal form. In such a situation, when the mean value of individual oscillators is constant corresponding to a fixed point type attractor, those oscillators behave randomly without any correlation. When the averaged dynamical variable between the oscillators is periodic corresponding to limit cycle type attractor, it suggests synchronized oscillatory states under the influence of noise as a result of cooperative phenomena.

In this paper, we propose a model of mean-field coupled excitable oscillators under the influence of the external Langevin noise. Satisfying self-averaging property, we obtain the time evolution of the order parameters for the system without approximations. Conducting bifurcation analyses, we show a bifurcation diagram for the order parameters of the system compared to numerical simulation with changes in noise strength. Part of this work has been briefly reported in the conference proceedings [10].

2. A Coupled Excitable Oscillator Model

To understand the effects of noise on synchronization in an infinitely many coupled excitable oscillator system exactly, let us consider the following form of stochastic differential equations [10]:

$$\frac{\mathrm{d}Z_i^{(x)}}{\mathrm{d}t} = -a^{(x)}Z_i^{(x)} + \frac{1}{N}\sum_{j=1}^{N} J^{(x)}F^{(x)}(\tilde{Z}_j^{(x)}) + I + \eta_i^{(x)}(t), \tag{1}$$

$$\frac{\mathrm{d}Z_i^{(y)}}{\mathrm{d}t} = \kappa(-a^{(y)}Z_i^{(y)} + \frac{1}{N}\sum_{j=1}^{N} J^{(y)}F^{(y)}(\tilde{Z}_j^{(y)})) + \eta_i^{(y)}(t), \tag{2}$$

where $\tilde{Z}_j^{(\mu)} = b^{(\mu,x)}Z_j^{(x)} + b^{(\mu,y)}Z_j^{(y)}$ $(i = 1,...,N)(\mu = x, y)$. $Z_i^{(\mu)}$ is a real-valued random dynamical variable of an oscillator and $a^{(\mu)}, b^{(\mu,\nu)}, J^{(\mu)}$ are constants, I is the applied current and $F^{(\mu)}(\cdot)$ is a coupling function. The parameter κ is introduced so that $Z_i^{(x)}$ become fast variables and $Z_i^{(y)}$ slow ones. To make an individual oscillator excitable, we specify $F^{(x)}(\cdot)$ and $F^{(y)}(\cdot)$ as nonlinear and linear functions as

$$F^{(x)}(Z) = Z\exp\left(-\frac{Z^2}{2}\right), \tag{3}$$

$$F^{(y)}(Z) = Z. \tag{4}$$

We postulate that the Langevin noise $\eta_i^{(\mu)}(t)$ is white Gaussian one, $\langle\eta_i^{(\mu)}(t)\rangle = 0$, $\langle\eta_i^{(\mu)}(t)\eta_j^{(\nu)}(t')\rangle = 2D^{(\mu)}\delta_{ij}\delta_{\mu\nu}\delta(t-t')$. In the absence of noise, Eqs. (1) and (2) with $N = 1$ recover 1-body deterministic oscillatory system which dynamical behavior is shown in Fig 1.

3. Nonlinear Fokker-Planck Equation Approach

The set of the Langevin equations (1) and (2) can be reduced to a single body equation as seen below in the thermodynamic limit $N \to \infty$. The mean-field coupling terms satisfy the self-average property, which is written by the empirical probability density $P(t, \mathbf{Z})$ $(\mathbf{Z}^T = (Z^{(x)}, Z^{(y)}))$ as

$$\langle F^{(\mu)}\rangle \equiv \int \mathrm{d}Z^{(x)}\mathrm{d}Z^{(y)}F^{(\mu)}(b^{(\mu,x)}Z^{(x)}+b^{(\mu,y)}Z^{(y)})P(t,\mathbf{Z}). \tag{5}$$

Then the system consisting of Eqs. (1) - (2) is indeed reduced to the single body dynamics $Z^{(x)}$ and $Z^{(y)}$ as

$$\frac{\mathrm{d}Z^{(x)}}{\mathrm{d}t} = -a^{(x)}Z^{(x)} + J^{(x)}\langle F^{(x)}\rangle + I + \zeta^{(x)}(t),$$
$$\frac{\mathrm{d}Z^{(y)}}{\mathrm{d}t} = \kappa(-a^{(y)}Z^{(y)} + J^{(y)}\langle F^{(y)}\rangle) + \zeta^{(y)}(t),$$

with white Gaussian noise $\zeta^{(\mu)}(t)$ as $\langle\zeta^{(\mu)}(t)\rangle = 0$, $\langle\zeta^{(\mu)}(t)\zeta^{(\nu)}(t')\rangle = 2D^{(\mu)}\delta_{\mu\nu}\delta(t-t')$. $\langle\cdot\rangle$ denotes the average over $P(t, \mathbf{Z})$. Thus, one obtains the NFPE for the empirical probability density corresponding to the above Langevin equations,

$$\begin{aligned}\frac{\partial}{\partial t}P(t, Z^{(x)}, Z^{(y)}) = &\\ -\frac{\partial}{\partial Z^{(x)}}&\left[-a^{(x)}Z^{(x)} + J^{(x)}\langle F^{(x)}\rangle + I - D^{(x)}\frac{\partial}{\partial Z^{(x)}}\right]P\\ -\frac{\partial}{\partial Z^{(y)}}&\left[\kappa(-a^{(y)}Z^{(y)} + J^{(y)}\langle F^{(y)}\rangle) - D^{(y)}\frac{\partial}{\partial Z^{(y)}}\right]P.\end{aligned} \tag{6}$$

A Gaussian probability density is a special solution of the NFPE (6). Since the H theorem [17] ensures that the probability density satisfying Eq. (6) converges to the Gaussian-form for sufficiently large times, it is enough that we only treat the Gaussian probability density as

$$P_{\mathrm{G}}(t, Z^{(x)}, Z^{(y)}) = \frac{1}{2\pi\sqrt{\det C_{\mathrm{G}}(t)}}\exp\left[-\frac{1}{2}\mathbf{s}_{\mathrm{G}}^T C_{\mathrm{G}}^{-1}(t)\mathbf{s}_{\mathrm{G}}\right],$$
$$\mathbf{s}_{\mathrm{G}}^T = (Z^{(x)} - \langle Z^{(x)}\rangle_{\mathrm{G}},\ Z^{(y)} - \langle Z^{(y)}\rangle_{\mathrm{G}}) \equiv (U^{(x)},\ U^{(y)}),$$
$$C_{\mathrm{G}\mu\nu}(t) = \langle s_\mu s_\nu\rangle_{\mathrm{G}},$$

where $\langle\cdot\rangle_{\mathrm{G}}$ denotes expectation over P_{G}. Then, the coupling term Eq. (5) is described only up to the second moments. Hence, one has a set of closed ordinary differential equations as

$$\frac{\mathrm{d}\langle Z^{(x)}\rangle_{\mathrm{G}}}{\mathrm{d}t} = -a^{(x)}\langle Z^{(x)}\rangle_{\mathrm{G}} + J^{(x)}\langle F^{(x)}\rangle_{\mathrm{G}} + I, \tag{7}$$

$$\frac{\mathrm{d}\langle Z^{(y)}\rangle_{\mathrm{G}}}{\mathrm{d}t} = \kappa(-a^{(y)}\langle Z^{(y)}\rangle_{\mathrm{G}} + J^{(y)}\langle F^{(y)}\rangle_{\mathrm{G}}), \tag{8}$$

$$\frac{\mathrm{d}\langle U^{(x)^2}\rangle_{\mathrm{G}}}{\mathrm{d}t} = -2a^{(x)}\langle U^{(x)^2}\rangle_{\mathrm{G}} + 2D^{(x)}, \tag{9}$$

$$\frac{\mathrm{d}\langle U^{(y)^2}\rangle_{\mathrm{G}}}{\mathrm{d}t} = -2\kappa a^{(y)}\langle U^{(y)^2}\rangle_{\mathrm{G}} + 2D^{(y)}, \tag{10}$$

$$\frac{\mathrm{d}\langle U^{(x)}U^{(y)}\rangle_{\mathrm{G}}}{\mathrm{d}t} = -(a^{(x)} + \kappa a^{(y)})\langle U^{(x)}U^{(y)}\rangle_{\mathrm{G}}, \tag{11}$$

where one has from Eqs. (3) - (4)

$$\langle F^{(x)}\rangle_{\mathrm{G}} = \frac{m^{(x)}}{(\sigma^2+1)^{3/2}}\exp\left[-\frac{m^{(x)^2}}{2(\sigma^2+1)}\right], \tag{12}$$

$$\langle F^{(y)}\rangle_{\mathrm{G}} = m^{(y)}, \tag{13}$$

and $m^{(\mu)} = b^{(\mu,x)}\langle Z^{(x)}\rangle_{\mathrm{G}} + b^{(\mu,y)}\langle Z^{(y)}\rangle_{\mathrm{G}}$, $\sigma^2 = b^{(x,x)^2}\langle U^{(x)^2}\rangle_{\mathrm{G}} + b^{(x,y)^2}\langle U^{(y)^2}\rangle_{\mathrm{G}}$. Note that $\langle U^{(x)}U^{(y)}\rangle_{\mathrm{G}} \to 0$, $\langle U^{(x)^2}\rangle_{\mathrm{G}} \to D^{(x)}/a^{(x)}$, and $\langle U^{(y)^2}\rangle_{\mathrm{G}} \to D^{(y)}/(\kappa a^{(y)})$ $(t \to \infty)$, implying that the external Langevin noise contributes to the dynamics through the variance. For simplicity, we assume the Langevin noise intensity $D^{(y)} = 0$ in what follows.

To understand nonlinear aspects of the dynamical system obtained by Eqs. (7)-(11) together with Eqs. (12) and (13), we conduct bifurcation analyses shown in Fig 2. The bifurcation of the order parameters for the system from fixed point type to limit cycle type attractor occurs with increasing noise intensity, which corresponds to the asynchronous to synchronous phase transition. Further increase of noise strength makes the individual oscillators to behave randomly. We can see the good agreement with numerical simulation calculated by directly solving the original Langevin equations.

4. Conclusion

We have introduced a mean-field model for coupled excitable oscillators to analyze noise-induced synchronization. An NFPE approach has enabled us to investigate the effects of noise on the system only by dealing with deterministic nonlinear dynamics. Taking the thermodynamic limit, the time evolution equations of the order parameters for the system have been derived without any approximations. A bifurcation diagram for both theoretically and numerically collective behavior of the oscillators with changes in noise intensity has been shown.

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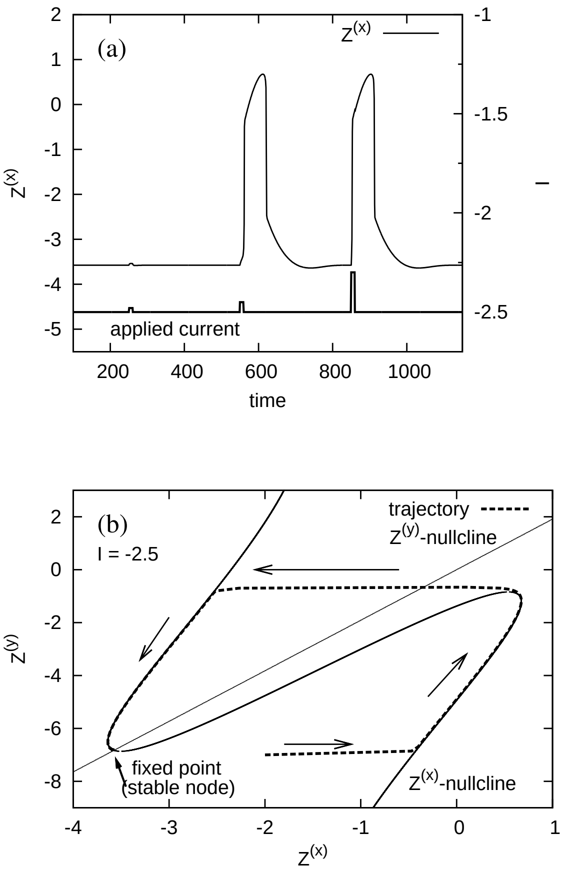

Figure 1: Excitable behavior of our model without noise. (a) The time evolution of $Z^{(x)}$. $Z^{(x)}$ gets excited only by large enough strength of the perturbation $\delta(t)$ given in the form of a short pulse at $t = 550.0, 850.0$: $I(t) = I + \delta(t)$. (b) Nullclines in the phase plane. $Z^{(x)}$ and $Z^{(y)}$ correspond to fast and slow variables, respectively. At $I = -2.5$, the trajectory converges to the stable node after sufficiently large times. The model parameter values are $a^{(x)} = 1.6875$, $a^{(y)} = 2.0925$, $b^{(x,x)} = 0.87750$, $b^{(x,y)} = -0.33750$, $b^{(y,x)} = 2.7000$, $b^{(y,y)} = -0.67500$, $J^{(x)} = 6.0000$, $J^{(y)} = 2.8350$, and $\kappa = 0.0068000$.

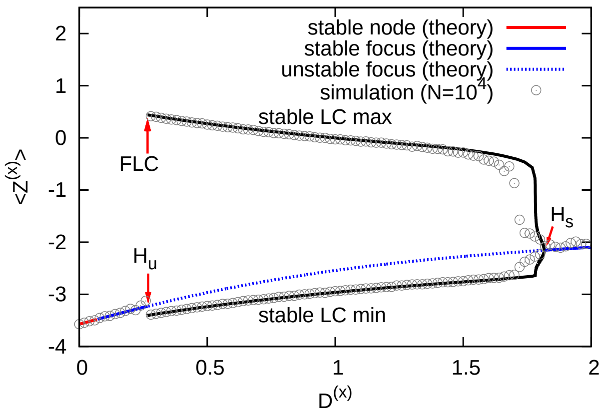

Figure 2: Bifurcation diagram of the system of coupled excitable elements under the influence of noise. Bifurcation analyses are conducted for the obtained order parameter system. Numerical simulation for the set of the original Langevin equations is also conducted using Euler-Maruyama method. The theoretical solid curves for the order parameters are good agreement with the numerical simulation, except for near the bifurcation point due to finite size effect. H_s and H_u, and FLC denote the supercritical and subcritical Hopf bifurcation, and fold limit cycle, respectively.