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Anomalous diffusion generated by quasiperiodically forced maps with strange nonchaotic attractors

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Abstract—The dispersion of trajectories in chaotic systems with spatial translational symmetry is referred to as deterministic diffusion. In a previous study, one of the authors showed that quasiperiodically forced systems with strange nonchaotic attractors (SNAs) produce anomalous diffusion characterized by nonlinear time evolutions of the variance. In the present study, we investigate anomalous diffusion generated by quasiperiodically forced maps with SNAs both numerically and experimentally. Due to SNAs, subdiffusion was observed in numerical simulations but in the experiment using an electronic circuit subdiffusion was not clearly observed due to the effect of noise. Nevertheless, we observed large fluctuations of local slope in the time evolution of the variance for the case of SNA.

1. Introduction

The one-dimensional random walk is a well known diffusion process. The variance (or mean square displacement) $\langle x_n^2\rangle$ of the random walkers grows linearly in time. Similarly, in a discrete time chaotic system, a partial sum $x_n = \sum_{i=1}^{n} v_i$ of chaotic variables v_i shows a diffusive process. The dispersion of the ensemble of x_n is called a *chaotic diffusion*. In typical chaotic systems, where the autocorrelation fuction $\langle v_n v_{n+\tau}\rangle$ decays exponentially fast, the variance $\langle x_n^2\rangle$ grows linearly in time. A type of diffusion with a linear growth of variance is called normal diffusion.

A strange nonchaotic attractor (SNA) is a geometrically strange attractor for which typical orbits have nonpositive Lyapunov exponents [1]. SNAs typically appear in quasiperiodically forced dynamical systems and possess intermediate properties between quasiperiodicity and chaos [2]. Mitsui and Aihara recently show that SNAs appear in several models of glacial cycles [3]. The aim of this study is to elucidate statistical properties of SNAs (especially, diffusion properties) from both theoretical and experimental points of view.

It is reported that the partial sum $x_n = \sum_{i=1}^{n} v_i$ of a variable v_i of an SNA does not show normal diffusion but anomalous diffusion, where the variance grows logarithmically $\langle x_n^2\rangle \sim \log n$ [4, 5] or sublinearly $\langle x_n^2\rangle \sim n^\alpha$ $(0 < \alpha < 1)$ [5]. In the present study, the such anomalous diffusion properties are shown for SNAs in a quasiperiodically forced map and an SNA observed in an experiment using an electronic circuit.

2. Subdiffusion due to antipersistence

Let us consider a diffusion process of "particles" on the real line. The position of a particle x_n at time $n = 0, 1, 2, ...$ is written as

$$x_n = x_0 + \sum_{k=0}^{n-1} v_k, \tag{1}$$

where v_n is a step size. Assume $\{v_n\}$ to be a *wide sense stationary process*, which satisfies $\langle v_n\rangle$ = constant and $\langle(v_n - \langle v_n\rangle)(v_{n+\tau} - \langle v_{n+\tau}\rangle)\rangle = \mathrm{c}(\tau)$. Here $\langle\cdot\rangle$ denotes an ensemble average, and the covariance $\mathrm{c}(\tau)$ is independent of n. The most basic quantities for the density distribution are the *mean displacement* $\langle x_n - x_0\rangle$ and the *variance*

$$\langle\sigma^2(n)\rangle = \langle(x_n - x_0 - \langle x_n - x_0\rangle)^2\rangle = \langle(x_n - x_0)^2\rangle - \langle x_n - x_0\rangle^2. \tag{2}$$

To characterize the temporal behavior of the variance, we use the *local scaling exponent* $\alpha(n)$ such that

$$\langle\sigma^2(2n)\rangle = 2^{\alpha(n)}\langle\sigma^2(n)\rangle. \tag{3}$$

The *average scaling exponent* in the time interval $[2^m, 2^M]$ is defined by

$$\overline{\alpha}_{m,M} = \frac{1}{M-m}\sum_{k=m}^{M-1}\alpha(2^k). \tag{4}$$

Then, we have $\langle\sigma^2(2^M)\rangle = 2^{\overline{\alpha}_{m,M}(M-m)}\langle\sigma^2(2^m)\rangle$. If the variance $\langle\sigma^2(n)\rangle$ grows roughly as a power law, $\overline{\alpha}_{m,M}$ is approximately-constant for large time difference $M-m$, namely,

$$\overline{\alpha}_{m,M} \simeq \overline{\alpha}. \tag{5}$$

If Eq. (5) holds for $M = \log_2 n$ and $m = 0$, the variance is written as

$$\langle\sigma^2(n)\rangle \sim n^{\overline{\alpha}} \quad (n \to \infty). \tag{6}$$

The diffusion is called *normal diffusion* if $\overline{\alpha} = 1$, and it is called *anomalous diffusion* if $\overline{\alpha} \neq 1$. Especially, *subdiffusion* corresponds to $0 < \overline{\alpha} < 1$, and *superdiffusion* corresponds to $1 < \overline{\alpha} < 2$.

The behavior of the variance $\langle\sigma^2(n)\rangle$ is related to the *correlation coefficient* $C(n)$ *between successive increments in* n *units of time*:

$$C(n) = \frac{\langle(x_{2n} - x_n - \langle x_{2n} - x_n\rangle)(x_n - x_0 - \langle x_n - x_0\rangle)\rangle}{[\langle(x_{2n} - x_n - \langle x_{2n} - x_n\rangle)^2\rangle\langle(x_n - x_0 - \langle x_n - x_0\rangle)^2\rangle]^{\frac{1}{2}}}. \tag{7}$$

Under the assumption of stationarity, the correlation coefficient $C(n)$ can be written as $C(n) = \langle\sigma^2(2n)\rangle/(2\langle\sigma^2(n)\rangle) - 1$. Using Eq. (3), the local scaling exponent $\alpha(n)$ is given as

$$\alpha(n) = 1 + \log_2[1 + C(n)]. \tag{8}$$

From Eqs. (4) and (8), the values of $\alpha(n)$ can be classified as follows: The negative correlation $-1/2 < C(n) < 0$ for $n = 2^k$ $(k = m, ..., M - 1)$, called *antipersistence*, means subdiffusion with $0 < \overline{\alpha}_{m,M} < 1$, the positive correlation $0 < C(n) < 1$ for $n = 2^k$ $(k = m, ..., M - 1)$, called *persistence*, means superdiffusion with $1 < \overline{\alpha}_{m,M} < 2$, and the absence of correlation $C(n) = 0$ for $n = 2^k$ $(k = m, ..., M - 1)$ means normal diffusion with $\overline{\alpha}_{m,M} = 1$.

If $\{v_t\}$ is ergodic, we have $\langle v_t\rangle = \overline{v_t}$ and $\langle v_t v_{t+n}\rangle = \overline{v_t v_{t+n}}$, where the right-hand-sides are the long time averages with respect to t. Then, the ensemble averaged variance $\langle\sigma^2(n)\rangle$ can be replaced by the time averaged variance $\overline{\sigma^2(n)} = \overline{(x_{t+n} - x_t)^2} - (\overline{x_{t+n} - x_t})^2$ along a typical trajectory, that is,

$$\langle\sigma^2(n)\rangle = \overline{\sigma^2(n)}.$$

3. Numerical study

Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, the unit circle. The quasiperiodically forced sine map $M : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ is given by [7, 4]

$$\begin{aligned} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= x_n + \frac{a}{2\pi}\sin(2\pi x_n) + \varepsilon\sin(2\pi\theta_n), \end{aligned}$$

where a and ε are parameters and ω is irrational. If we consider x_n on $\mathbb{T}^1$ by taking modulo 1, M reduces to the quasiperiodically forced sine circle map $\tilde{M}$ on $\mathbb{T}^2$.

When $\tilde{M}$ has an attractor, the attractor can be classified using the conditional Lyapunov exponent $\lambda = \lim_{n\to\infty}(1/n)\ln|\partial x_n/\partial x_0|$ and the phase sensitivity exponent ν [6], which measures the sensitivity of the attractor with respect to changes in the phase θ_n. The attractor is a two-frequency quasiperiodic attractor if $\lambda < 0$ and $\nu = 0$, a three-frequency quasiperiodic attractor if $\lambda = 0$, an SNA if $\lambda < 0$ and $\nu > 0$, or a chaotic attractor if $\lambda > 0$ [7, 4].

In what follows, we set $\varepsilon = 2.5$ and $\omega = (\sqrt{5} - 1)/2$, the golden mean. In a previous study, we showed that $\tilde{M}$ has a quasiperiodic attractor for $a < 0.850\,23$, an SNA for $0.850\,23 < a < a_c$ $(a_c \approx 1.828\,7)$, and a chaotic attractor for $a > a_c$. Figure 1(a) shows the SNA for $a = 1.7$. The phase sensitivity of SNA causes the diffusion of x_n with different initial phases θ_0 (see Fig. 1(b)).

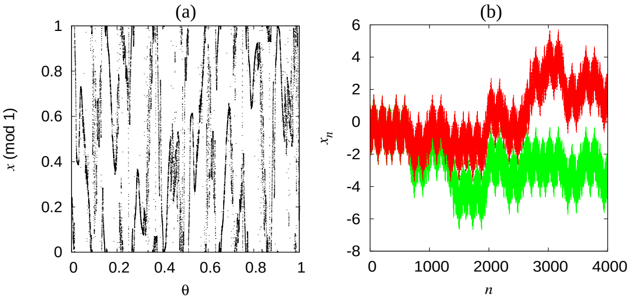

Figure 1: (a) SNA for $a = 1.7$. (b) Separation of trajectories x_n with slightly different initial phases $\theta_0 = 0$ and 10^{-4}.

We calculate the variance $\langle\sigma^2(n)\rangle$ given by Eq. (2) for an ensemble of 5000 trajectories initially distributed uniformly over the region $-0.5 \le x \le 0.5$, $0 \le \theta < 1$. For each trajectory, the first 5000 iterations were discarded considering them transients, and the 5001st state was set to (x_0, θ_0). Since M is symmetric with respect to the transformation $(x, \theta) \to (-x, \theta + 1/2 \pmod 1)$, we have $\langle x_n - x_0\rangle = 0$ and $\langle\sigma^2(n)\rangle = \langle(x_n - x_0)^2\rangle$.

In a parameter region corresponding to SNAs ($1.65 \lesssim a < a_c$), the variance $\langle\sigma^2(n)\rangle$ grows roughly as a power law, as shown in Fig. 2(a), although there exist deviations from strict power laws, which are rather conspicuous when a is far from the transition point $a = a_c$. Figure 2(b) shows the exponent $\overline{\alpha}$ as a function of a, which is estimated in the interval $[2^{20}, 2^{30}]$ $(\approx [10^6, 10^9])$. This result shows that M roughly exhibits subdiffusion with $0 < \overline{\alpha} < 1$ near the transition point to chaos. In the parameter region corresponding to chaotic attractors ($a > a_c$), we observe normal diffusion with $\overline{\alpha} = 1$ [see Figs. 2(a) and 2(b)]. We suppose that the crossover from subdiffusion to normal diffusion occurs at $a = a_c$ because the exponent α at $a = a_c$ approaches 1 as m and $(M - m)$ increase.

Figures 3(a) and 3(b) show the correlation coefficient $C(n)$ and the local scaling exponent $\alpha(n)$ for $a = 1.7$ (SNA) and $a = 2.0$ (chaos), respectively. It is shown that the subdiffusion with $0 < \alpha(n) < 1$ is caused by the antipersistence $-1/2 < C(n) < 0$ for large n. On the other hand, the normal diffusion with $\alpha(n) = 1$ is caused by the decay of the correlation $C(n) = 0$ for large n. Therefore, the crossover from subdiffusion to normal diffusion results from the loss of antipersistence owing to the transition from strange nonchaotic to chaotic dynamics.

It should be mentioned that the diffusion process is ergodic. As shown in the inset of Fig. 2(a), the values of $\langle\sigma^2(n)\rangle$ and $\overline{\sigma^2(n)}$ are identical.

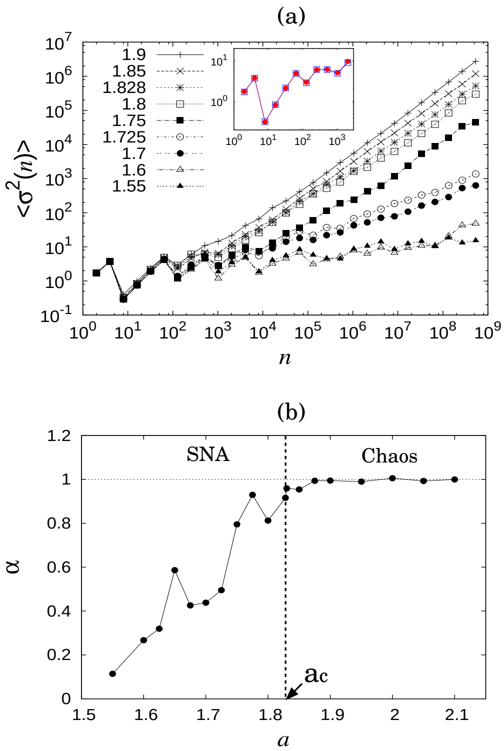

Figure 2: (a) Evolution of the variance $\langle\sigma^2(n)\rangle$ for different values of a below and above $a_c \approx 1.828\,7$. Data are plotted at times $n = 2^k$ ($k = 1, 2, ..., 29$). Inset shows the values of $\langle\sigma^2(n)\rangle$ and $\overline{\sigma^2(n)}$, which are almost identical. (b) Scaling exponent $\overline{\alpha}$ as a function of a. The exponent $\overline{\alpha}$ is obtained from the variance in the time interval $[2^{20}, 2^{30}]$ ($\approx [10^6, 10^9]$).

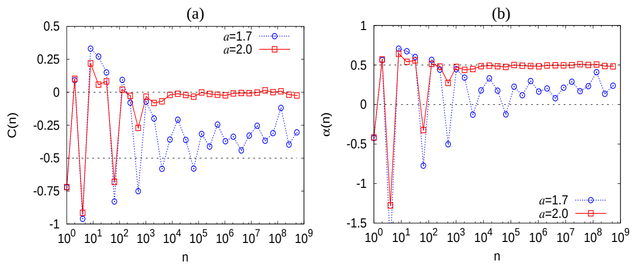

Figure 3: (a) Correlation coefficient $C(n)$ between successive increments in n units of time for $a = 1.7$ (blue open circle) and $a = 2.0$ (red open square). The value of $C(n)$ is calculated by using Eq. (7). (b) Local scaling exponent $\alpha(n)$ for $a = 1.7$ (blue open circle) and $a = 2.0$ (red open square). The value of $\alpha(n)$ is calculated by using Eq. (3). In both panels, data are plotted at times $n = 2^k$ ($k = 0, 1, ..., 29$).

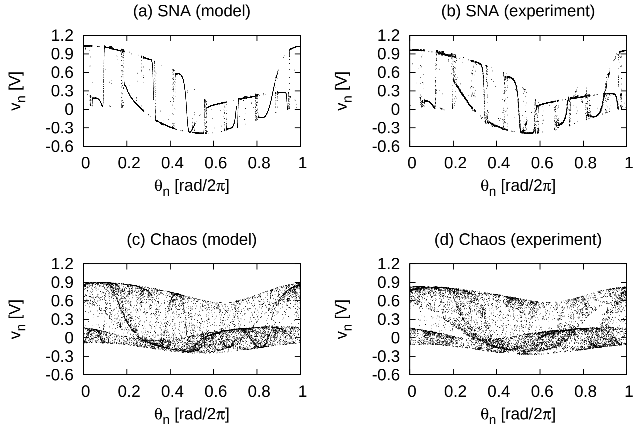

Figure 4: Attractor of map (9). (a) SNA simulated for $b = 0.13$ V. (b) SNA experimentally obtained for $b = 0.13$ V. (c) Chaotic attractor simulated for $b = 0.05$ V. (d) Chaotic attractor experimentally obtained for $b = 0.05$ V.

4. Experimental Study

In our previous study [8], we generated SNAs and chaotic attractors in a switched-capacitor integrated circuit [9]. In the present study, we analyze diffusion properties of the SNAs and chaotic attractors obtained in [8]. The circuit was designed to implement the chaotic neuron model [10] under quasiperiodic forcing as follows:

$$\begin{aligned} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ v_{n+1} &= kv_n + c f(v_n + b\cos 2\pi\theta_n) + a + D\xi_n, \end{aligned} \tag{9}$$

where v_n is the internal state of the neuron model, θ_n is the phase of quasiperiodic forcing with amplitude b and irrational frequency $\omega = (\sqrt{5} - 1)/2$, and c is a scale factor. We consider the additive white Gaussian noise $D\xi_n$ to account for experimental results, where D is the amplitude of noise. The function $f(\cdot)$ is a monotonically decreasing continuous function determined by the physical properties of MOSFETs in the circuit. Refer to [8] for the details of the experiment. For $b = 0.13$ V, we obtained an SNA in the simulation without noise $D = 0$ V (Fig. 4(a)) and an SNA in the experiment (Fig. 4(b)). For $b = 0.05$ V, we obtained a chaotic attractor in the simulation without noise $D = 0$ V (Fig. 4(c)) and a chaotic attractor in the experiment (Fig. 4(d)).

In the experiment, we observed a single time series of v_n of length $N = 10^6$ for each attractor. A diffusion process x_n is again defined as Eq. (1), where $x_0 = 0$. For each x_n, we calculate the time averaged variance $\overline{\delta^2(n)}$. Because of the finite length of the time series, $N = 10^6$, the values of the variance $\overline{\delta^2(n)}$ are reliable only for $n \lesssim O(10^2)$. Figure 5 shows the behavior of the variance $\overline{\delta^2(n)}$ for the SNAs and chaotic attractors obtained in the simulations

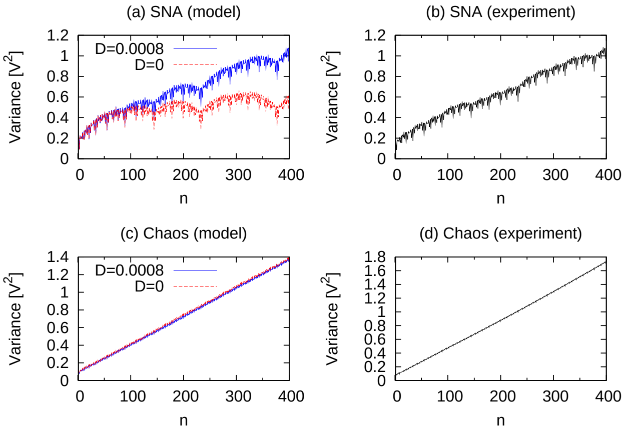

Figure 5: Time evolution of variance $\overline{\delta^2(n)}$. (a) The variance $\overline{\delta^2(n)}$ for the SNA simulated in the absence of noise ($b = 0.13$ V and $D = 0$ V) (blue solid line) and for the SNA simulated under in the presence of noise ($b = 0.13$ V and $D = 0.8 \times 10^{-3}$ V) (red dashed line). The difference between the two lines is not trivial because the variance purely due to noise is $D^2 n < 2.6 \times 10^{-4}$; (b) The variance $\overline{\delta^2(n)}$ for the experimentally obtained SNA ($b = 0.13$ V). (c) The variance $\overline{\delta^2(n)}$ for the chaotic attractor simulated in the absence of noise ($b = 0.05$ V and $D = 0$ V) (blue solid line) and for the chaotic attractor simulated under in the presence of noise ($b = 0.05$ V and $D = 0.8 \times 10^{-3}$ V) (red dashed line); (d) The variance $\overline{\delta^2(n)}$ for the experimentally obtained chaotic attractor ($b = 0.05$ V).

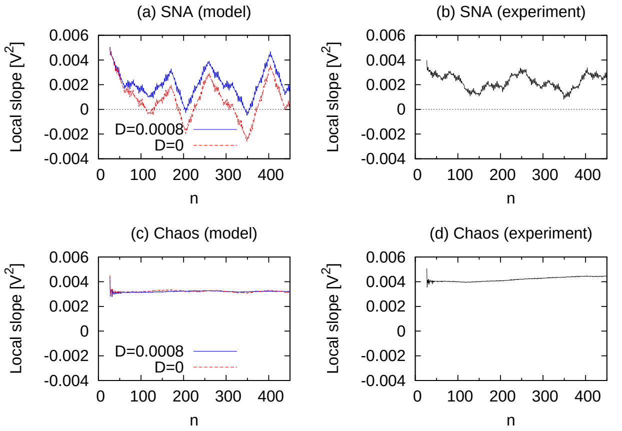

Figure 6: Local slope of variance $LS(n) = (\overline{\delta^2(n+28)} - \overline{\delta^2(n-27)})/55$. The panels from (a) to (d) in this figure correspond to those in Fig. 5, respectively. All the panels have the same axis ranges for comparison.

and the experiment. To characterize the evolution of the variance, we use a local slope of the variance $LS(n) = (\overline{\delta^2(n+28)} - \overline{\delta^2(n-27)})/55$, as shown in Fig. 6. For the SNA obtained in the simulation without noise $D = 0$ V, the growth of the variance $\overline{\delta^2(n)}$ seems roughly sublinear (see red dashed line in Fig. 5(a)), but for the SNA obtained in the experiment, the growth of the variance $\overline{\delta^2(n)}$ seems roughly linear in the time scale of $O(10^2)$ (see solid line in Fig. 5(b)). Such a roughly linear growth of $\overline{\delta^2(n)}$ is reproduced in the simulation with noise $D = 0.8 \times 10^{-3}$ V (see blue solid line in Fig. 5(a)). However, for all the cases of SNAs, we observe large fluctuations of the local slope $LS(n)$ as shown in Figs. 6(a) and 6(b). On the other hand, for the chaotic attractors, the variance $\overline{\delta^2(n)}$ grows linearly as shown in Figs. 5(c) and 5(d), and the local slope $LS(n)$ is almost constant (see Figs. 6(c) and 6(d)).

5. Summary

We investigated anomalous diffusion generated by quasiperiodically forced maps with SNAs both numerically and experimentally. Due to SNAs, subdiffusion was observed in numerical simulations but in the experiment using an electronic circuit subdiffusion was not clearly observed due to the effect of noise. Nevertheless, we observed large fluctuations of local slope in the time evolution of the variance for the case of SNA.

Acknowledgments

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References

[1] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke, Physica D, vol.13, p.261, 1984.

[2] A. Prasad, S. S. Negi, R. Ramaswamy, Int. J. Bifurcation Chaos Appl. Sci. Eng., vol.11, p.291, 2001.

[3] T. Mitsui, K. Aihara, Climate Dynamics, DOI: 10.1007/s00382-013-1793-x, published online.

[4] U. Feudel, J. Kurths, A. Pikovsky, Physica D, vol.88, p.176, 1995.

[5] T. Mitsui, Phys. Rev. E, vol.83, 066212, 2011.

[6] A. Pikovsky, U. Feudel, Chaos, vol.27, p.253, 1995.

[7] M. Ding, C. Grebogi, E. Ott, Phys. Rev. A, vol.39, p.2593, 1989.

[8] S. Uenohara, T. Mitsui, Y. Hirata, T. Morie, Y. Horio, K. Aihara, Chaos, vol.23, 023110, 2013.

[9] Y. Horio, K. Aihara, O. Yamamoto, IEEE Trans. Neural Networks, vol.14, p.1393, 2003.

[10] K. Aihara, T. Takebe, M. Toyoda, Phys. Lett. A, vol.144, p.333, 1990.