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Abstract—Weak noise smooths out fractals in a chaotic state space and introduces a maximum attainable resolution to its structure. The balance of noise and deterministic stretching/contraction in each neighborhood introduces local invariants of the dynamics that can be used to partition the state space. We study the local discrete-time evolution of a density in a two-dimensional hyperbolic state space, and use the asymptotic eigenfunctions for the noisy dynamics to formulate a new state space partition algorithm.

1. Motivation and outline

Chaotic systems' main feature is their high sensitivity to initial conditions. That makes direct numerical integration of the equations difficult and often calls for alternative methods for the evaluation of long-time averages of observables, such as decay of correlations, diffusion coefficients, energy spectra, or escape rates [7]. To properly weigh these averages, one needs to understand which regions of the state space are more or less relevant for the dynamics, in other words make a *partition* [3]. Invariants of the dynamics such as unstable periodic oribts have been successfully used to partition the state space [1].

However, noise, modelled by stochastic variables, erases periodic orbits. One has to look for new invariants. For that reason, we previously have studied [4, 2] the evolution of *densities* of trajectories and determined eigenfunctions of the local Fokker-Planck operator in the vicinity of the deterministic periodic orbits. The eigenfunctions are then used to partition the state space. All that was done in discrete time in one dimension. In order to develop a similar algorithm in higher dimensions, the first step is again to study the evolution of densities in the neighborhood of the periodic points of the deterministic system.

In the present contribution we focus on the asymptotic evolution in two dimensions, forward and backward in time of a noiseless hyperbolic map (sect. 2), to which we successively add weak, uncorrelated, isotropic noise (sect. 3). In both cases the densities asymptotically align with the unstable (stable) direction of the monodromy matrix when iterated forward (backward) in time. We finally use our results to propose a definition of a *neighborhood* for an optimal partition of the state space.

2. Deterministic evolution

We start by reviewing the deterministic evolution of densities and observables in the neighborhood of a fixed point $\mathbf{x}_0$ of the two-dimensional map $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. We assume that the fixed point is hyperbolic, i.e., that the Jacobian matrix evaluated at the fixed point,

$$M_{ij}(\mathbf{x}_0) = \left.\frac{\partial f_i(\mathbf{x})}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x_0}} , \tag{1}$$

has eigenvalues $|\Lambda_s| < 1$, $|\Lambda_u| > 1$.

Consider the simplest example, a map

$$\mathbf{f}(\mathbf{x}) = (\Lambda_s x, \Lambda_u y) \tag{2}$$

that is contracting along the x-axis and expanding along the y-axis. Now consider a density of trajectories $\rho(\mathbf{x})$, for instance a Gaussian placed around the fixed point of $\mathbf{f}(\mathbf{x})$, and apply the Perron-Frobenius operator [1] to it:

$$\mathcal{L}\rho(\mathbf{x}) = \int d\mathbf{z}\,\delta\left(\mathbf{x}-\mathbf{f}(\mathbf{z})\right)\rho(\mathbf{z}) = \frac{1}{|\Lambda_u \Lambda_s|}\rho\left(\frac{x}{\Lambda_s}, \frac{y}{\Lambda_u}\right), \tag{3}$$

so that, after n iterations,

$$\mathcal{L}^n\rho(\mathbf{x}) = \frac{1}{|\Lambda_u|^n}\frac{\rho(\frac{x}{\Lambda_s^n}, \frac{y}{\Lambda_u^n})}{|\Lambda_s|^n} . \tag{4}$$

One can see this as a density, which is losing mass by a factor of $|\Lambda_u|^{-1}$ at each iteration. This expression can be renormalized by a factor of $|\Lambda_u|^n$, when taking the limit $n \to \infty$. If the initial density is a normalized Gaussian $\rho(\mathbf{x}) \propto \exp\left[-(x^2+y^2)/2\sigma^2\right]$, we obtain

$$\lim_{n\to\infty} |\Lambda_u|^n \mathcal{L} = \lim_{n\to\infty} \frac{\exp\left[-\frac{x^2}{2(\sigma\Lambda_s^n)^2}\right]}{\sqrt{2\pi\sigma^2\Lambda_s^{2n}}} = \delta(x) \tag{5}$$

meaning the limiting density is supported on the y-axis, the *unstable manifold* of the fixed point of the map.

Of course this was the simplest possible example, given that the contracting and expanding directions of the fixed point are already separated by the coordinates. This is not the case in general, and one needs to do something different from what we just described. We will follow Rugh's formalism [8] for a general two-dimensional map $\mathbf{f}(\mathbf{x})$ with a hyperbolic fixed point $\mathbf{x}_0$: the equation $f_y(x_i, y_i) = y_f$ has a unique solution, which we can call $\phi_s(x_i, y_f)$, which is analytic and a contraction. On the other hand, one can define

$$\phi_u(x_i, y_f) = f_x\left(x_i, \phi_s(x_i, y_f)\right) \tag{6}$$

and then rewrite $\mathbf{f}$,

$$\mathbf{f}\left(x_i, \phi_s(x_i, y_f)\right) = \left(\phi_u(x_i, y_f), y_f\right), \tag{7}$$

in terms of the *pinning coordinates* (x_i, y_f), that is the contracting coordinate of the initial point, x_i and the expanding coordinate of the final point, y_f. It is important to remark that both $\phi_u(x_i, y_f)$ and $\phi_s(x_i, y_f)$ are contractions on their supports [8] . In particular, for fixed x_i,

$$\lim_{n\to\infty} \phi_s^n(x_i, y_f) = W^s(x_i) \tag{8}$$

with $W^s(x_i)$ such that $(x_i, W^s(x_i))$ parametrizes the stable manifold of the map $\mathbf{f}(\mathbf{x})$. Similarly,

$$\lim_{n\to\infty} \phi_u^n(x_i, y_f) = W^u(y_f) \tag{9}$$

where $\left(W^u(y_f), y_f\right)$ defines the unstable manifold of $\mathbf{f}(\mathbf{x})$.

Now we will obtain again the limit (5) for the evolution of a density carried by the Perron-Frobenius operator. We will study the evolution (4) inside a space average of an observable $a(\mathbf{x})$.

$$\langle a\rangle_n = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} d\mathbf{x}\, a(\mathbf{x})\,[\mathcal{L}^n\rho(\mathbf{x})] \tag{10}$$

As we will show, the support of the observable inside the average corresponds to the support of the mapped density. Eq. (10) is equivalent to a more familiar expression for the space average, which we can easily write by letting $\mathcal{L}^n$ act on its left on the observable a, in which case it becomes the Koopman operator $\mathcal{K}$ [1],

$$\begin{aligned}\int_{\mathcal{M}} d\mathbf{x} a(\mathbf{x})\,[\mathcal{L}^n\rho(\mathbf{x})] &= \int_{\mathcal{M}} d\mathbf{x}\,[\mathcal{K}^n a(\mathbf{x})]\,\rho(\mathbf{x}) \\ &= \int_{\mathcal{M}} d\mathbf{x}\, a\,(\mathbf{f}^n(\mathbf{x}))\,\rho(\mathbf{x}) \end{aligned}\tag{11}$$

We now change the coordinates in the last integral according to the transformation (7):

$$\int_I dx_i dy_f a\left(\phi_u^n(x_i, y_f), y_f\right)\rho\left(x_i, \phi_s^n(x_i, y_f)\right)\det\left(\partial_2\phi_s^n(x_i, y_f)\right) \tag{12}$$

where I is the new domain of integration and $\det\left(\partial_2\phi_s^n(x_i, y_f)\right)$ is the determinant of the Jacobian of the change of coordinates in the integral. For $n \to \infty$ one gets, according to (8) and (9),

$$\int_I dx_i dy_f a\left(W^u(y_f), y_f\right)\rho\,(x_i, W^s(x_i))\det\left(\partial_2\phi_s^\infty(x_i, y_f)\right). \tag{13}$$

We can see that the observable a ends up being supported on the *unstable manifold* of the map. Intuitively, the initial observable stretches and contracts respectively along the unstable and stable manifolds, so that it asymptotically survives on the only region of the state space (the unstable manifold) where it cannot be crushed by the contraction. In the separable case (2), the average (10) is proportional to

$$\begin{aligned}\int dxdy\, a(x, y)\mathcal{L}^n\rho(x, y) \to \tfrac{1}{|\Lambda_u|^n}\int dxdy\, a(x, y)\delta(x) = \\ \tfrac{1}{|\Lambda_u|^n}\int dy\, a(0, y).\end{aligned} \tag{14}$$

In other words, knowing the support of the observable a inside the average is equivalent to knowing the support of the time-forward evolved density ρ (in this case the y–axis, cf. (5)).

Now we consider the time-backward evolution, described by the Koopman operator $\mathcal{K}$:

$$\begin{aligned}\int_{\mathcal{M}} d\mathbf{x}\, a(\mathbf{x})\,[\mathcal{K}^n\rho(\mathbf{x})] \to \\ \int_I dx_i dy_f\, a\,(x_i, W^s(x_i))\,\rho\left(W^u(y_f), y_f\right)\det\left(\partial_2\phi_s^\infty(x_i, y_f)\right)\end{aligned} \tag{15}$$

where we switched to pinning coordinates as in (13), with ρ and a inverted with respect to the previous case, and then took the limit $n \to \infty$. This time the observable a is asymptotically supported on the *stable manifold*.

Analogous results are found for a periodic point $\mathbf{x}_a$ of a map $\mathbf{f}(\mathbf{x})$, for it can be regarded as the fixed point of the iterated map $\mathbf{f}^{n_p}(\mathbf{x})$, with n_p period of the cycle. The time-forward (backward) evolution aligns observables and thus densities to the unstable (stable) eigenvector of the *monodromy* matrix

$$M_{ij}^{n_p}(\mathbf{x}_a) = \left.\frac{\partial f_i^{n_p}(\mathbf{x})}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_a} \tag{16}$$

evaluated at the periodic point $\mathbf{x}_a$.

3. Adding noise

We now add weak noise to the map $\mathbf{f}(\mathbf{x})$. In the vicinity of any point x_a, Gaussian densities are mapped forward in time by the Fokker-Planck operator $\mathcal{L}_{FP}$ [2, 6]

$$\begin{aligned}\rho_{a+1}(z_{a+1}) &= \frac{1}{C_a}\int [dz_a]\, e^{-\frac{1}{2}(z_{a+1}-M_a z_a)^\top \frac{1}{\Delta}(z_{a+1}-M_a z_a) - z_a{}^\top \frac{1}{Q_a} z_a} \\ &= \frac{1}{C_{a+1}}\, e^{-\frac{1}{2} z_{a+1}{}^\top \frac{1}{Q_{a+1}} z_{a+1}},\end{aligned} \tag{17}$$

where we defined local coordinates $z_a = x - x_a$. Here the noise is described by the symmetric and positive definite diffusion tensor Δ. If the density is a Gaussian distribution, we can recast the problem in terms of its covariance matrix [2]:

$$Q_{a+1} = M_a Q_a M_a{}^\top + \Delta\,. \tag{18}$$

The long-time limit is given by the fixed-point condition $Q_a = Q_{a+1}$, valid when the dynamics is contracting (M has all eigenvalues $|\Lambda_i| < 1$). This condition states that the covariance matrix must be invariant under the combined action of the deterministic contraction and expansion by weak noise after one time step. Let S be the matrix which diagonalizes M. If we make the further transformations $Q \to S^{-1}Q\big(S^{-1}\big)^\top \equiv \hat{Q}$ and $\Delta \to S^{-1}\Delta\big(S^{-1}\big)^\top \equiv \hat{\Delta}$, the solution to the fixed-point condition for contracting maps reads [2]

$$\hat{Q}_{ij} = \frac{1}{1-\Lambda_i\Lambda_j}\hat{\Delta}_{ij}. \tag{19}$$

We obtain the time-backward evolution by taking the adjoint of the operator in (17). Like before, an equation is derived for the mapping of the covariance matrix:

$$M_a Q_a {M_a}^\top = Q_{a+1} + \Delta. \tag{20}$$

If the deterministic dynamics is expanding (M has all eigenvalues $|\Lambda_i| > 1$), (20) becomes a fixed point condition by setting $Q_a = Q_{a+1}$. We find that

$$\hat{Q}_{ij} = \frac{1}{\Lambda_i\Lambda_j - 1}\hat{\Delta}_{ij}, \tag{21}$$

where we applied the same diagonalization transformation by means of the matrix S.

Typically in a chaotic system the matrix M has both contracting and expanding directions, so that neither the solution given by (19) nor by (21) applies. In what follows we study the evolution of the covariant matrix both forward and backward in time, looking for an asymptotic limit. We start forward in time, iterating (18) in the neighborhood of a fixed point of $\mathbf{f}(\mathbf{x})$:

$$Q_n = \Delta + M\Delta M^\top + M^2\Delta(M^2)^\top + \cdots + M^n Q_0 (M^n)^\top . \tag{22}$$

As in the deterministic case, let us first see how the Q's map when M and the initial Q_0 are diagonal: each matrix element Q_{ii} obeys the sum (22), which diverges for $|\Lambda_i| > 1$, so that Q_{ii}^{-1} vanishes. On the other hand, it converges to $Q_{ii} \to \Delta_{ii}/(1-\Lambda_i^2)$ [4] if $|\Lambda_i| < 1$. As a result, the axes of the Gaussian $e^{-z^\top Q^{-1} z}$ asymptotically survive in the stable directions only, while the whole density is supported along the unstable directions, like in the deterministic separable case (cf. (5)). In two dimensions, the asymptotic density looks like the Gaussian-shaped tube in figure 1(d).

We will now explain the asymptotic evolution of the axes of the ellipsoid the Gaussian is supported on when M is not diagonal, mimicking an argument of Ott's [5]. First of all, if the noise is isotropic, $\Delta(x) = 2D\mathbf{1}$, the diffusion tensor Δ is replaced by a scalar diffusion constant D that can be factored from the sum (22). Consider a vector v:

$$M^n(M^n)^\top v = M^n \sum_j a_j \Lambda_j^n \hat{e}'_j \sim M^n a_u \Lambda_u^n \hat{e}'_u \tag{23}$$

where we first wrote v in terms of the eigenvectors $\hat{e}'_j$ of $M^\top$, then we applied $(M^n)^\top$ to the same vector and finally we observed that the result asymptotically aligns to the direction $\hat{e}'_u$ of the most unstable eigenvalue of $M^\top$, Λ_u. Let

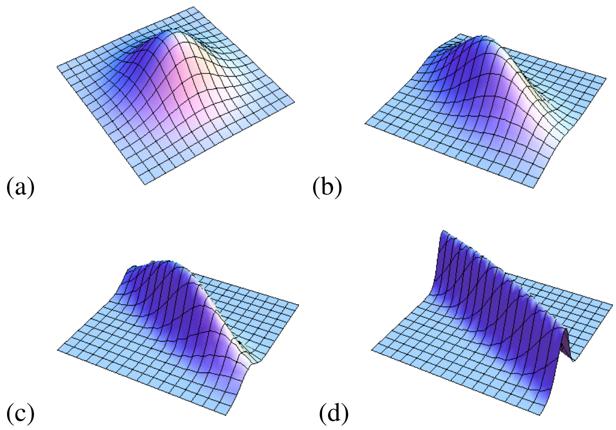

Figure 1: Evolution of a Gaussian density centered at the fixed point of a hyperbolic map: (a) initial bump; (b) after one iteration; (c) after two iterations; (d) asymptotic limit.

us now write $M^n = S\hat{\Lambda}^n S^{-1}$, with $S = (\hat{e}_1, ..., \hat{e}_u, ..., \hat{e}_N)$, and $S^{-1} = (\hat{e}'_1, ..., \hat{e}'_u, ..., \hat{e}'_N)^\top$. So now

$$\begin{aligned} M^n \Lambda_u^n \hat{e}'_u = S\hat{\Lambda}^n S^{-1} \Lambda_u^n \hat{e}'_u \propto \\ S\hat{\Lambda}^n (\hat{e}'_u \cdot \hat{e}'_1, \ldots, 1, \ldots, \hat{e}'_u \cdot \hat{e}'_N)^\top = \\ S\left(\Lambda_1^n \hat{e}'_1 \cdot \hat{e}'_u, \ldots, \Lambda_u^n, \ldots, \Lambda_N^n \hat{e}'_N \cdot \hat{e}'_u\right)^\top \end{aligned} \tag{24}$$

When $n \to \infty$ the uth component of the vector weighs above all others, making the result proportional to

$$S(0, ..., 1, ..., 0)^\top = \hat{e}_u . \tag{25}$$

Thus any vector is eventually stretched and rotated toward *the most unstable direction* of M. It is straightforward to show that the last term in the sum (22), $M^n Q_0 (M^n)^\top$, asymptotically behaves likewise. Consequently, Q_∞ and thus Q_∞^{-1} are also aligned with the most unstable eigenvector of the monodromy matrix M. Numerics (figure 1) help us visualize the result in two dimensions, where M has one stable and one unstable directions: an initial isotropic Gaussian develops into a 'tube', infinitely extended along the unstable manifold of M, and having a Gaussian section in the orthogonal direction, due to the balance of noise and deterministic contraction.

One can repeat the above reasoning when applying the adjoint Fokker-Planck operator $\mathcal{L}_{FP}^\dagger$. In this case we invert (20) (the unknown being Q_a), and iterate it n times to get

$$Q_{-n} = M^{-n} Q_0 (M^{-n})^\top + M^{-n}\Delta(M^{-n})^\top + ... + M^{-1}\Delta(M^{-1})^\top . \tag{26}$$

This is similar to (22), except the monodromy matrix is inverted, so that stable and unstable eigenvalues are swapped, and the argument (23)-(25) results in any vector being stretched and rotated toward *the most stable direction* of M.

The observations made for a fixed point of the map can be extended to a periodic orbit of arbitrary period n_p. We start again from (22):

$$Q_a = \Delta + M_{a-1}\Delta(M_{a-1})^\top + \cdots + M^{n_p}_{a-n_p} Q_{a-n_p} (M^{n_p}_{a-n_p})^\top \,. \quad (27)$$

We define

$$\Delta_{p,a} = \Delta + M_{a-1}\Delta(M_{a-1})^\top + \cdots + M^{n_p-1}_{a-n_p+1}\Delta(M^{n_p-1}_{a-n_p+1})^\top, \quad (28)$$

and then the asymptotic evolution of the covariance matrix Q around the periodic point x_a can be written as

$$Q_{p,a} = M_{p,a} Q_a M_{p,a}{}^\top + \Delta_{p,a} \,, \qquad (29)$$

where $M_{p,a} = M^{n_p}_a$, the latter defined in (16). The problem reduces to the previous case, which shows that the leading eigenvector of the asymptotic covariance matrix (or the major axis of the ellipsoid) aligns to the most unstable (stable) eigenvector of $M_{p,a}$ in the time-forward (backward) evolution.

Following the technique we used in one dimension [4], we define the neighborhood $\mathcal{M}_a$ of the periodic point x_a as the intersection of the supports (within a 1σ confidence) of the ground-state local eigenfunctions of $\mathcal{L}_{FP}$ and $\mathcal{L}^\dagger_{FP}$. We use these regions to cover the non–wandering set of the system, starting with the periodic orbits of the shortest period, and increasing the period until neighborhoods significantly overlap (see figure 2(d)).

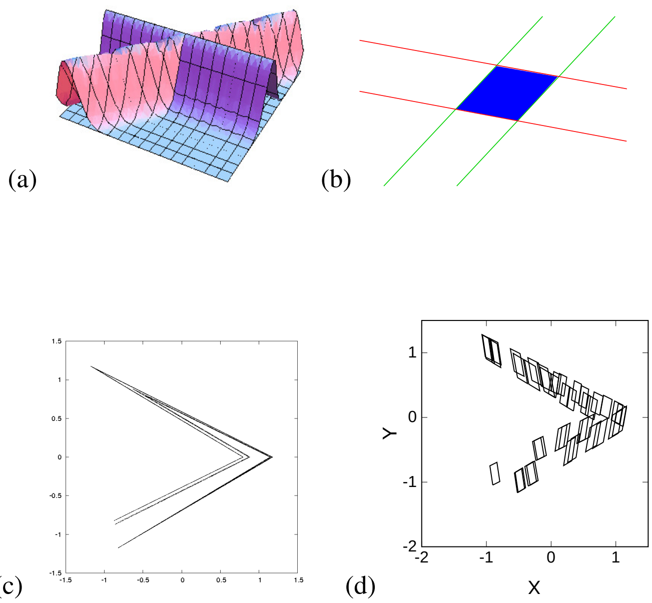

Figure 2: (a) The ground-state eigenfunctions of $\mathcal{L}_{FP}$ and of its adjoint $\mathcal{L}^\dagger_{FP}$, both operator linearized around the same fixed point; (b) our definition of partition interval in two dimensions: take the local densities in (b), cut off their supports at 1σ and take their intersections. The Lozi attractor [1], (c) noiseless, and (d) noisy, covered with neiborhoods from all periodic points up to length six.

4. Summary

We have studied the asymptotic evolution of Gaussian densities of trajectories in the neighborhoods of hyperbolic periodic points, first in a deterministic map and then in the presence of weak, uncorrelated, isotropic noise. We investigated both time-forward and -backward dynamics. The latter was realized by means of the adjoint of the evolution operator. As it turns out, the densities asymptotically align with the most unstable (stable) direction of the monodromy matrix when iterated forward (backward) in time. Using both asymptotic densities, we proposed a definition for a *neighborhood* which should then be used to partition the state space.

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References

[1] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay. *Chaos: Classical and Quantum*. Niels Bohr Institute, Copenhagen, 2013. ChaosBook.org.

[2] P. Cvitanović and D. Lippolis. Knowing when to stop: How noise frees us from determinism. In M. Robnik and V. G. Romanovski, editors, *Let's Face Chaos through Nonlinear Dynamics*, pages 82–126, Melville, New York, 2012. Am. Inst. of Phys. arXiv:1206.5506.

[3] B. Kitchens, editor. *Symbolic Dynamics. One-sided, Two-sided, and Countable Markov shifts*. Springer, Berlin, 1998.

[4] D. Lippolis and P. Cvitanović. How well can one resolve the state space of a chaotic map? *Phys. Rev. Lett.*, 104:014101, 2010. arXiv:0902.4269.

[5] E. Ott. *Chaos and Dynamical Systems*. Cambridge University Press, 2002.

[6] H. Risken. *The Fokker-Planck Equation*. Springer, New York, 1996.

[7] D. Ruelle. *Statistical Mechanics, Thermodynamic Formalism*. Addison-Wesley, Reading, MA, 1978.

[8] H. H. Rugh. The correlation spectrum for hyperbolic analytic maps. *Nonlinearity*, 5:1237, 1992.