

Stable Parameter Estimation of Compound Wishart Distribution for Polarimetric SAR Data Modeling

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Abstract—This paper investigates the parameter estimation of compound Wishart distribution for statistical modeling of polarimetric synthetic aperture radar data. We show that the recently proposed method of matrix log-cumulant may lead to non-invertible equations and cause unstable performances. In order to overcome such difficulty, we proposed a Bayesian-based method to re-estimate the log-cumulants. Simulation experiment demonstrates that the proposed algorithm provides both improved and stabler results. Finally, we present an application of the new method for texture analysis of the Germany F-SAR polarimetric data.

I. INTRODUCTION

Polarimetric synthetic aperture radar (POL SAR) is an important microwave remote sensing system. It is able to provide a high-resolution map of the ground terrains and at the same time polarimetric scattering matrices. However, since POL SAR is a coherent system, the acquired scattering matrix is subject to a noise-like phenomenon, often called speckle in radar community [1]. Although strictly speaking, speckle is not really noise but a useful signal, it becomes a nuisance in the context of single-dataset acquisition and should be removed for quantitative applications.

POL SAR data are characterized by both polarimetric and spatial information. The former is affected by speckle and the latter is related to the spatial pattern of the scene. In order to statistically model their joint behaviors, a product distribution has been popularly used [2]. It says that the measured observable is the multiplication of one Wishart-distributed random matrix and one positive random variable, which independently account for the speckle structure and texture variation (see Section II-B). Under this scheme, several compound distributions have been derived. Particularly, the K-Wishart distribution has been proved useful in a number of applications such as image classification or segmentation [3].

Clearly, the accuracy of applying the compound Wishart distribution depends on the accuracy of parameter estimation. Recently, based on the matrix-variate Mellin transformation, Anfinson et al. [2] proposed a new parameter estimation method using matrix log-cumulants. It appears to be promising because of the incorporation of full polarimetric information. However, as we will point out, this method may sometimes suffer from non-invertible problem and so unstable performances. This paper proposes a solution to overcome such difficulty. It is organized as follows. In Section II, a brief introduction of POL SAR data and the associated compound Wishart distribution is provided. In Section III, we describe

the parameter estimation using the method of matrix log-cumulant and propose the improved algorithm. In Section IV, experiments results with simulated and real POL SAR data are given. Section V is the conclusion.

II. POL SAR DATA AND COMPOUND WISHART DISTRIBUTION

A. POL SAR Data

In the horizontal/vertical (H/V) polarization basis, the POL SAR system measures, for each resolution cell, the scattering matrix:

$$\mathbf{S} = \begin{bmatrix} S_{HH} & S_{VH} \\ S_{HV} & S_{VV} \end{bmatrix}, \quad (1)$$

where S_{AB} represents the scattering coefficient in the A-receive/B-transmit channel. Under the backscattering reciprocal condition, we have $S_{HV} = S_{VH}$ such that (1) can be equally written by the vector:

$$\mathbf{k} = \begin{bmatrix} S_{HH} \\ \sqrt{2}S_{HV} \\ S_{VV} \end{bmatrix}, \quad (2)$$

Both \mathbf{S} and \mathbf{k} represent the single-look data format of the POL SAR measurement. The multi-look covariance matrix is defined as:

$$\mathbf{Z} = \frac{1}{L} \sum_{i=1}^L \mathbf{k}_i \mathbf{k}_i^H \quad (3)$$

where \mathbf{k}_i is the i -th independent look and \mathbf{k}_i^H denotes its conjugate transpose. According to (3), \mathbf{Z} is a 3×3 Hermitian and positive-definite matrix. In this paper, we focus on statistical modeling of such kind of data.

B. Product Model

The product model for the multi-look covariance matrix is given by [2]:

$$\mathbf{Z} = \gamma \mathbf{W}, \quad (4)$$

where \mathbf{W} is a Hermitian and positive-definite random matrix and γ is a positive random variable that is independent of \mathbf{W} . \mathbf{W} contains the speckle and polarimetric information and has a probability density function (pdf) of the Wishart distribution [4]:

$$p_{\mathbf{W}}(\mathbf{W}) = \frac{L^{Ld}}{\Gamma_d(L)} \frac{|\mathbf{W}|^{L-d}}{|\Sigma|^d} \exp[-\text{Tr}(L\Sigma^{-1}\mathbf{W})], \quad (5)$$

where d is the dimension of \mathbf{W} which equals 3 as in this paper; $\Gamma_d(L) = \pi^{d(d-1)/2} \prod_{i=0}^{d-1} \Gamma(L-i)$ is the multivariate generalization of the gamma function; $\Sigma = \mathbb{E}(\mathbf{W})$ is the mean covariance matrix. On the other hand, γ is a unit-mean positive random variable which characterizes textures of the scene. Its pdf can be set flexibly but the gamma distribution [5] as given in (6) has proved to be both useful and tractable.

$$p_\gamma(\gamma) = \frac{\nu^\nu}{\Gamma(\nu)} \gamma^{\nu-1} \exp(-\nu\gamma). \quad (6)$$

As a result of (4)–(6), the pdf of the observed covariance matrix \mathbf{Z} becomes [4]:

$$p_{\mathbf{Z}}(\mathbf{Z}) = \frac{2(L\nu)^{\frac{\nu+Ld}{2}} |\mathbf{Z}|^{L-d} [\text{Tr}(\Sigma^{-1}\mathbf{Z})]^{\frac{\nu-Ld}{2}}}{\Gamma_d(L)\Gamma(\nu) |\Sigma|^L} \times K_{\nu-Ld} \left[2\sqrt{L\nu \text{Tr}(\Sigma^{-1}\mathbf{Z})} \right], \quad (7)$$

where $K_\nu(\cdot)$ represents the modified Bessel function of the second kind. The above distribution is often called the compound K-Wishart distribution due to its generalization of the K-distribution in the single-variable case [2].

III. PARAMETER ESTIMATION OF K-WISHART DISTRIBUTION

According to (7), it can be seen that the compound K-Wishart distribution contains three parameters: L , Σ , and ν . In SAR statistics, L is often called the equivalent number of looks (ENL) indicating the speckle level of the multi-look data. It can be conveniently estimated from a homogeneous (non-textured) area [6] and maintains the same for a given system. Hence throughout this paper, L is assumed to be a known constant. Then the remaining task is to estimate Σ and ν .

A. Estimation of Mean Covariance Matrix

Suppose $\mathbf{Z}_i (i = 1, 2, \dots, n)$ are identically and independently distributed (i.i.d.) K-Wishart samples. According to (4), it is easy to obtain:

$$\mathbb{E}(\mathbf{Z}) = \mathbb{E}(\gamma \cdot \mathbf{W}) = \mathbb{E}(\gamma) \cdot \mathbb{E}(\mathbf{W}) = \Sigma. \quad (8)$$

where the fact that γ is a unit-mean variable independent of \mathbf{W} is used. From (8) it is clear that Σ can be straightforwardly estimated by:

$$\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n \mathbf{Z}_i. \quad (9)$$

B. Estimation of Shape Parameter

In [2], Anfinsen et al. proposed that by the matrix-variate Mellin transform, the shape parameter ν of the K-Wishart distribution can be related to the second-order matrix log-cumulant as follows:

$$\psi^{(1)}(\nu) = \frac{\hat{\kappa}_2\{\mathbf{Z}\} - \sum_{i=0}^{d-1} \psi^{(1)}(L-i)}{d^2}, \quad (10)$$

where $\psi^{(1)}(\cdot)$ is the trigamma function and the second-order log-cumulant is estimated by:

$$\hat{\kappa}_2\{\mathbf{Z}\} = \frac{1}{n} \sum_{i=1}^n (\ln |\mathbf{Z}_i|)^2 - \left(\frac{1}{n} \sum_{i=1}^n \ln |\mathbf{Z}_i| \right)^2. \quad (11)$$

Inversion of ν from (10) can be numerically accomplished by, e.g., the Newton method. However, since $\psi^{(1)}(\nu) > 0$ for all $\nu > 0$ [7], the solution to (10) exists if and only if

$$\hat{\eta} = \hat{\kappa}_2\{\mathbf{Z}\} - \sum_{i=0}^{d-1} \psi^{(1)}(L-i) > 0. \quad (12)$$

Unfortunately, this condition cannot be always satisfied due to the estimation uncertainty in (11). The situation becomes especially worse with small samples. For example, simulation by the true parameters of $L = 3$, $\nu = 10$, and

$$\Sigma = \begin{bmatrix} 11.9 & -2.5 + 1.0i & -0.8 - 1.0i \\ -2.5 - 1.0i & 3.4 & 0.2 + 0.3i \\ -0.8 + 1.0i & 0.2 - 0.3i & 1.3 \end{bmatrix} \quad (13)$$

indicates that with 49 independent samples, around 15% of the cases lead to the invertible equation of (10). This phenomenon necessitates to modify $\hat{\eta}$ in order to ensure its non-negativeness. The simplest method is to threshold it to zero whenever it becomes negative. It means forcing $\nu = \infty$ while the K-Wishart distribution degenerates to the standard Wishart distribution. However, such treatment may lead to very unstable results, as will be seen in Section IV-A.

In order to overcome the aforementioned problem, we propose to re-estimate the posterior mean of $\hat{\eta}$ with Bayesian method. Specifically, the posterior pdf of $\hat{\eta}$ is:

$$p(\eta_m | \hat{\eta}) = \frac{p(\hat{\eta} | \eta_m) p(\eta_m)}{p(\hat{\eta})}, \quad (14)$$

where η_m stands for the mean of $\hat{\eta}$. Clearly, in order to impose the non-negative constraint, a reasonable choice for $p(\eta_m)$ is the uniform distribution:

$$p(\eta_m) = \frac{1}{b}, 0 \leq \eta_m \leq b, \quad (15)$$

where b is an arbitrarily large positive number. Next we turn the attention to the conditional pdf $p(\hat{\eta} | \eta_m)$. Considering (11) and (12), we may approximate $p(\hat{\eta} | \eta_m)$ with a Gaussian distribution, an assumption justified by the central limit theorem:

$$p(\hat{\eta} | \eta_m) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(\hat{\eta} - \eta_m)^2}{2\sigma^2} \right]. \quad (16)$$

Consequently, in order to determine the functional form of $p(\hat{\eta} | \eta_m)$ it is necessary to obtain σ^2 , i.e., the variance of $\hat{\eta}$ (or equivalently, the variance of $\hat{\kappa}_2\{\mathbf{Z}\}$). This can be accomplished by a moment approach [8]. Specifically, if we let $z_i = \mathbf{Z}_i (i = 1, 2, \dots, n)$, the variance of $\hat{\kappa}_2\{\mathbf{Z}\}$ can be estimated by [8]:

$$\hat{\sigma}^2 = \left(\frac{1}{n} - \frac{2}{n^2} \right) \hat{\xi}_z^4 + \left(\frac{4}{n^2} - \frac{1}{n} \right) \hat{\sigma}_z^4. \quad (17)$$

where $\hat{\sigma}_z^2$ and $\hat{\xi}_z^4$ represent the estimation of the variance and kurtosis of z_i which are respectively given by:

$$\hat{\sigma}_z^2 = \frac{1}{n} \sum_{i=1}^n \left(z_i - \frac{1}{n} \sum_{j=0}^n z_j \right)^2, \quad (18)$$

$$\hat{\xi}_z^4 = \frac{1}{n} \sum_{i=1}^n \left(z_i - \frac{1}{n} \sum_{j=0}^n z_j \right)^4. \quad (19)$$

Finally, we calculate the posterior mean:

$$\bar{\eta}_m = \int \eta_m p(\eta_m | \hat{\eta}) d\eta_m. \quad (20)$$

Since we do not have any prior knowledge of b other than $b > 0$, we allow $b \rightarrow +\infty$ in (20) and perform the integration to obtain:

$$\bar{\eta}_m = \hat{\eta} + \frac{\exp\left(-\frac{\hat{\eta}^2}{2\hat{\sigma}^2}\right)}{\sqrt{2\pi}\Phi\left(\frac{\hat{\eta}}{\hat{\sigma}}\right)} \cdot \hat{\sigma}. \quad (21)$$

where $\Phi(\cdot)$ is the cumulative function of the standard normal distribution. It can be verified that $\eta_m > 0$ for any $\hat{\eta} \in \mathbb{R}$ and $\hat{\sigma} \in \mathbb{R}^+$. Therefore, instead of solving (10) we solve (22) as below for the shape parameter.

$$\psi^{(1)}(\nu) = \frac{\eta_m}{d^2}. \quad (22)$$

To sum, the parameter estimation algorithm for the K-Wishart distribution is re-stated in Algorithm 1.

Algorithm 1 Parameter Estimation of K-Wishart Distribution

- 1: **input:** $\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n$
 - 2: $\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n \mathbf{Z}_i$
 - 3: $\hat{\kappa}_2\{\mathbf{Z}\} = \frac{1}{n} \sum_{i=1}^n (\ln |\mathbf{Z}_i|)^2 - \left(\frac{1}{n} \sum_{i=1}^n \ln |\mathbf{Z}_i|\right)^2$
 - 4: $\hat{\eta} = \hat{\kappa}_2\{\mathbf{Z}\} - \sum_{i=0}^{d-1} \psi^{(1)}(L-i)$
 - 5: $\hat{\sigma}_z^2 = \frac{1}{n} \sum_{i=1}^n \left(\ln |\mathbf{Z}_i| - \frac{1}{n} \sum_{j=1}^n \ln |\mathbf{Z}_j| \right)^2$
 - 6: $\hat{\xi}_z^4 = \frac{1}{n} \sum_{i=1}^n \left(\ln |\mathbf{Z}_i| - \frac{1}{n} \sum_{j=1}^n \ln |\mathbf{Z}_j| \right)^4$
 - 7: $\hat{\sigma}^2 = \left(\frac{1}{n} - \frac{2}{n^2}\right) \hat{\xi}_z^4 + \left(\frac{4}{n^2} - \frac{1}{n}\right) \hat{\sigma}_z^4$
 - 8: $\bar{\eta}_m = \hat{\eta} + \left\{ \exp\left(-\frac{\hat{\eta}^2}{2\hat{\sigma}^2}\right) / \left[\sqrt{2\pi}\Phi\left(\frac{\hat{\eta}}{\hat{\sigma}}\right) \right] \right\} \cdot \hat{\sigma}$
 - 9: $\psi^{(1)}(\nu) = \bar{\eta}_m / d^2 \Rightarrow \hat{\nu}$
 - 10: **output:** $\hat{\Sigma}, \hat{\nu}$
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IV. EXPERIMENTAL RESULTS

A. Simulation Evaluation

In this subsection, simulated K-Wishart samples are generated to evaluate the proposed parameter estimation algorithm. The sample size is fixed to 49 (corresponding to a 7×7 moving window if adaptive estimation is required in POLSAR imagery). We choose $L = 3$ and Σ given in (13) as true parameters. Using both the original and proposed methods, the bias and standard deviation of $\hat{\nu}$ are shown in Fig. 1 and Fig. 2 when its true values varies from 5 to 50. It should be noted that by the original method, we mean estimation of the shape parameter directly from (10) and those cases when $\hat{\eta} < 0$ are discarded. From Fig. 1 and Fig. 2, large performance fluctuations can be observed for the original method whereas a significantly improved bias and standard deviation with much stabler behavior are seen for the proposed method.

The reason behind the performance differences in Fig. 1 and Fig. 2 can be explained by (21). Especially, the second term

on the right hand side of (21) may be considered as a variance stabilizer. It reduces the uncertainty of $\hat{\eta}$ due the introduction of the prior distribution. Such reduction propagates through (22) and finally leads to a stabler solution of $\hat{\nu}$.

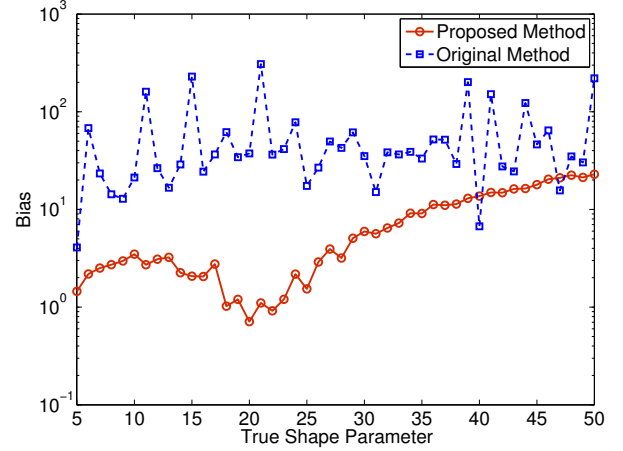


Fig. 1. Bias of the estimated shape parameter.

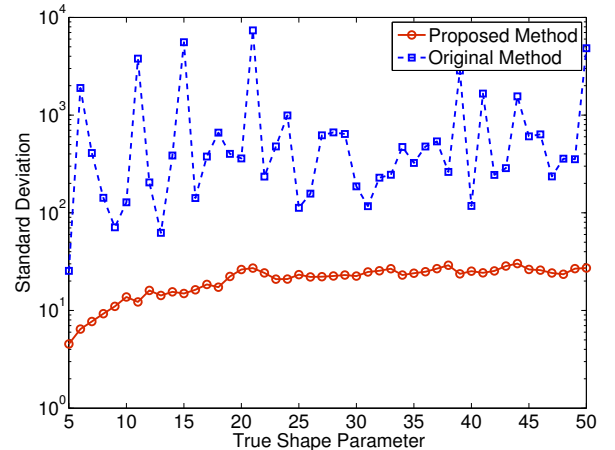


Fig. 2. Standard deviation of the estimated shape parameter.

B. Application with Real POLSAR Data

We provide one application of the proposed parameter estimation algorithm for texture analysis of real POLSAR images. The data used here is acquired by the Germany F-SAR system at the S-band. Fig. 3 displays the image color-coded in Pauli-basis. Multi-look preprocessing has been performed by combining 3×3 pixels to formulate the covariance matrix as given by (3). Therefore, the nominal number of looks is nine but it should be kept in mind that the ENL can be smaller than this value due to the oversampling of the data. Nonetheless, we will for the moment assume that $L = 9$. Later we will see how to obtain a more accurate of the ENL estimation by using the texture analysis result.

We assume that the multi-look covariance data follows the K-Wishart distribution. The shape parameter of the K-Wishart distribution ν is then adaptively estimated by using a 7×7 moving window. Fig. 4 shows the map of the estimated shape parameters using Algorithm 1 for each position of the image. Comparing Fig. 3 and Fig. 4, it is easy to see that high values are identified in homogeneous areas which indicates negligible texture effects whereas in urban and forested areas, low values are found, revealing the rich texture information therein.



Fig. 3. Pauli-basis color display of F-SAR data.



Fig. 4. Map of shape parameter estimation (in log-scale).

By thresholding the shape parameter homogeneous areas can be identified. For example, those areas where $\nu > 5L$ can be considered as homogeneous and we may use the pixels therein to refine the ENL estimation results obtained by [9]. Fig. 5 shows the histogram of the locally estimated ENL within the areas satisfying $\nu > 5L$. We see that the mode of such histogram approximately corresponds to 6.1. It is this value that is the true ENL of the synthesized multi-look data.

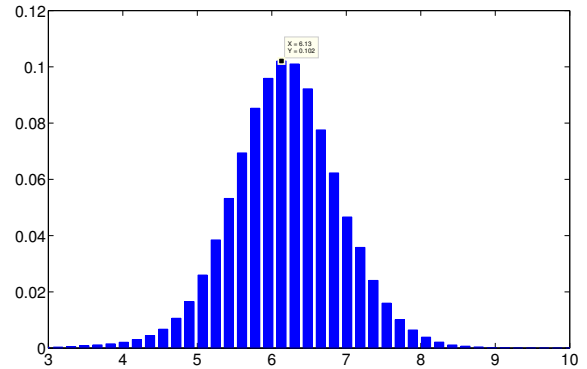


Fig. 5. Histogram of local ENL estimates in homogeneous areas.

V. CONCLUSION

In this paper, we have proposed a modified method of matrix log-cumulant for parameter estimation of the K-Wishart distribution that is significant in statistical POLSAR data modeling. The new approach applies Bayesian estimation of the second-order log-cumulant which ensures an always invertible equation for the shape parameter. Importantly, using simulated K-Wishart samples we have shown that the new method presents improved and stabler performance than the original one. Finally, we have demonstrated a promising application of the proposed algorithm for texture analysis of the real POLSAR imagery. It is found that with the extra texture information, a more accurate ENL estimation can be obtained.

REFERENCES

- [1] F.T. Ulaby, R.K. Moore, and A. K. Fung, *Microwave Remote Sensing: Active and Passive, vol. II, Radar Remote Sensing and Surface Scattering and Emission Theory*, Inc., Dedham, Massachusetts, 1986
- [2] S.N. Anfinson and T. Eltoft, "Application of the matrix-variate Mellin transform to analysis of polarimetric radar images," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, pp. 2281-2295, 2011.
- [3] A.P. Doulgeris, S.N. Anfinson, and T. Eltoft, "Automated non-Gaussian clustering of polarimetric synthetic aperture radar images," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, pp. 3665-3676, 2011
- [4] J.S. Lee and E. Pottier, *Polarimetric Radar Imaging: From Basics to Applications*, CRC Press, 2009.
- [5] C. Oliver and S. Quegan, *Understanding synthetic aperture radar images*. SciTech Publishing, 2004.
- [6] Y. Cui, G. Zhou, J. Yang, and Y. Yamaguchi, "Unsupervised estimation of the equivalent number of looks in SAR images," *IEEE Geosci. Remote Sens. Letters*, vol. 8, pp. 710-714, 2011.
- [7] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, and C.W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.
- [8] D.A. Abraham and A.P. Lyons, "Reliable methods for estimating the K-distribution shape parameter," *IEEE J. Oceanic Eng.*, vol. 35, pp. 288-301, 2010.
- [9] S.N. Anfinson, A.P. Doulgeris and T. Eltoft, "Estimation of the equivalent number of looks in polarimetric synthetic aperture radar imagery," *IEEE Trans. Geosci. Remote Sens.*, vol. 47, 3795-3809, 2009.