STATISTICAL ASPECTS OF RADAR POLARIMETRY

Ernst Lüneburg[1] (1932 - 2004) and Wolfgang-Martin Boerner[2]
[1]EML Consultants
Georg-Schmid Weg 4, Oberpfaffenhofen, D-82234 WESSLING, Germany
T/F: + [49] (8153) 1405/952983
E-mail: e.lueneburg@t-online.de
[2]University of Illinois at Chicago, ECE/CSN,
M/C 154, 900 W. Taylor Street, SEL-W (607) – 4210, CHICAGO, IL/USA 60607-7018
T&F: + [1] (312) 996-5480
E-mail: boerner@ece.uic.edu

1. Introduction.

In radar and in optical polarimetry there exist essentially two different methods to characterize the polarimetric scattering properties of plane fully polarized electromagnetic waves by randomly distributed targets using second-order multivariate statistics: the Kennaugh matrix and the covariance matrix formulations. They are generally considered to be different and independent, although formally they involve the same second-order multivariate moments. The Kennaugh approach is used for finding solutions for maximal and minimal power transfer between the transmitting and receiving antenna whereas the covariance matrix analysis is used for entropy and variance considerations and for the generation of uncorrelated random variables. Second-order statistics involve complete information for multivariate Gaussian distributions and in general provide sufficient information for sub- and supergaussian distributions. In the following we consider in particular backscatter radar polarimetry and Boerner et al [1] and Mott [2] for a detailed account.

In the far field of any scatterering object the elements of the 2×2 polarimetric scattering matrix S in a linear appropriately chosen linear transmit and receive basis for the domain and range of the scattering operator are correlated random variables where t stands for time or ensemble value and first and second index denote receiver transmitter polarization basis vectors

$$S(t)=\begin{bmatrix} S_{x_r x_t}(t) & S_{x_r y_t}(t) \\ S_{y_r x_t}(t) & S_{y_r y_t}(t) \end{bmatrix}$$

Assuming ergodicity ensemble averages can be replaced by time averages and will be denoted by sharp brackets $<\ldots>$. For the sake of simplicity we take $<S(t)>=0$ which implies the removal of the means. A coherently scattering target, also called a point or a deterministic target, will be denoted by the same symbol but without round brackets. Using a common polarization basis system for the interesting and important case of backscattering (the BSA convention) the Sinclair backscattering matrix is symmetric due to reciprocity for all instants of time as discussed by Lueneburg [3], [4].

2. Formulation of the Covariance and Kennaugh Matrices

The correlated random matrix element are arranged as a column vector, the so-called target feature vector. This can be done in different ways. The simplest one is to use the lexicographical ordering by taking the complete set of basis matrices

$$\Psi_L=\sqrt{2}\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\ \begin{bmatrix}0&1\\0&0\end{bmatrix},\ \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}.$$

and form the column vector of **correlated** random variables

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$$\vec{k}_L(t) = \text{trace}\{S(t)\cdot \text{vec}(\Psi_L)\} = \begin{bmatrix} k_1^L(t) \\ k_2^L(t) \\ k_3^L(t) \\ k_4^L(t) \end{bmatrix} = \begin{bmatrix} S_{HH}(t) \\ S_{VH}(t) \\ S_{HV}(t) \\ S_{VV}(t) \end{bmatrix} = \text{vec}\, S(t)$$

with the polarimetrically invariant Euclidean norm

$$< \| \vec{k}_L(t) \|^2 > = 2\,\text{span}\, S(t) = 2\,\text{span}\, U^T S(t) U =$$
$$= 2\left\{ < \| S_{HH}(t) \|^2 > + < \| S_{VH}(t) \|^2 > + < \| S_{HV}(t) \|^2 > + < \| S_{VV}(t) \|^2 > \right\}.$$

By use of the Hermitian traceless Pauli spin matrices

$$\Psi_P \equiv \{\sigma_0, \vec{\sigma}\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

that obey the commutation relations $[\sigma_i, \sigma_j] \equiv \sigma_i \sigma_j - \sigma_j \sigma_j = 2\varepsilon_{ijk}\sigma_k$.

The Levi-Civita-Symbol ε_{ijk} denotes the 3-rank tensor 3:

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if two indexes are equal} \\ 1, & \text{if } i,j,k \text{ are an even permutation} \\ -1, & \text{if } i,j,k \text{ are an oddd permutation} \end{cases}.$$

We thus obtain the target vector

$$\vec{k}_P(t) = \text{trace}\{S(t)\cdot \Psi_P\} = \begin{bmatrix} S_{HH} + S_{VV} \\ S_{HV} + S_{VH} \\ j(S_{HV} - S_{VH}) \\ S_{HH} - S_{VV} \end{bmatrix}$$

with the same invariant norm $\| \vec{k}_P(t) \|^2 = \| \vec{k}_L(t) \|^2$. The target vectors $\vec{k}_L(t)$ and $\vec{k}_P(t)$ are related by the unitary transformation V

$$\vec{k}_P(t) = V\,\vec{k}_L(t), \qquad V = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -j & j & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

By time or ensemble averaging one can form the following expressions from the feature vector $\vec{k}_L(t)$ or $\vec{k}_P(t)$

$$C_L = < \vec{k}_L(t)\vec{k}_L^{\dagger}(t) > \quad \text{and} \quad C_P = < \vec{k}_P(t)\vec{k}_P^{\dagger}(t) >.$$

In the literature C_L is called the (lexicographic) covariance matrix whereas C_P is named Pauli's coherency matrix and the whole analysis based upon these matrices is known as (Cloude's) target decomposition theory.

Covariance, correlation, coherency, coherence matrices: second order moments of multivariate joint probability density function. Sufficient only for (complex) Gaussian pdf. In general resource to methods of independent component analysis (ICA).

Target decomposition is a special application of general Principal Component Analysis (PCA) to radar polarimetry, Pearson 1901, Hotelling 1933.

The matrices C_L and C_P are Hermitian positive semidefinite. They are related by unitary similarity $C_L = V^{-1} C_P V$ and, hence, the same eigenvalues. Now, the target vector $\vec{k}_P$ cannot be obtained from $\vec{k}_L$ by going over to a different polarization basis for the Sinclair matrix S or the Jones matrix J.

The covariance matrix C_L, say, can be unitarily diagonalized

$$W^{-1} C_L W = \Lambda \equiv \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3, \lambda_4], \quad 0 \le \lambda_4 \le \lambda_3 \le \lambda_2 \le \lambda_1$$

or with $W = [\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4]$

$$C_L \vec{x}_i = \lambda_i \vec{x}_i, \quad \vec{x}_i^{\dagger} \vec{x}_j = \delta_{ij} \quad (i, j = 1,2,3,4).$$

The components $\vec{z}_i(t)$ $(i = 1,2,3,4)$ of the random vector

$$\vec{Z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix} = W^{\dagger} \vec{k}_L(t) = \begin{bmatrix} \vec{x}_1^{\dagger} \vec{k}_L(t) \\ \vec{x}_2^{\dagger} \vec{k}_L(t) \\ \vec{x}_3^{\dagger} \vec{k}_L(t) \\ \vec{x}_4^{\dagger} \vec{k}_L(t) \end{bmatrix}$$

are called the principal components (PC's). They are uncorrelated (but not necessarily independent) and their variance is equal to the corresponding eigenvalue of C_L

$$< \vec{z}_i(t) \vec{z}_j^{*}(t) > = x_i^{\dagger} < \vec{k} \vec{k}_L^{\dagger} > \vec{x}_j = x_i^{\dagger} C_L \vec{x}_j = \lambda_j \delta_{ij}.$$

The vectors $\vec{x}_i$ $(i = 1,2,3,4)$ are called the vectors of coefficients or loadings for the ith PC.

The spectral decomposition of C_L reads $C_L = W \Lambda W^{\dagger} = \sum_{i=1}^{4} \lambda_i \vec{x}_i \vec{x}_i^{\dagger}$.

The four 4×4 matrices $\vec{x}_i \vec{x}_i^{\dagger}$ all have rank 1 and can be interpreted as deterministic 2×2 point targets S_i

$$S_i = \begin{bmatrix} x_{i1} & x_{i3} \\ x_{i2} & x_{i4} \end{bmatrix} \quad \text{with} \quad \mathrm{vec} S_i = \vec{x}_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \\ x_{i4} \end{bmatrix} \quad (i = 1,2,3,4).$$

Reversing the vec-operation yields the target decomposition

$$S(t) = \begin{bmatrix} S_{HH}(t) & S_{HV}(t) \\ S_{VH}(t) & S_{VV}(t) \end{bmatrix} = \sum_{i=1}^{4} \begin{bmatrix} x_{i1} & x_{i3} \\ x_{i2} & x_{i4} \end{bmatrix} z_i(t) = \sum_{i=1}^{4} S_i z_i(t)$$

where the basic target matrices S_i are deterministic point targets with $\mathrm{span} S_i = 1$. The symmetry of the Sinclair matrix $S(t)$ for backscattering implies also the symmetry of all basic targets S_i. It should be stressed that the random variables $z_i(t)$ are uncorrelated and that the basic targets S_i are orthonormal in the sense

$$(\mathrm{vec} S_i)^{\dagger} \mathrm{vec} S_j = \sum_{j=1}^{4} | x_{ij} |^2 \delta_{ij} = |\vec{x}_i|^2 \; \delta_{ij} = \delta_{ij}.$$

The average Graves' power matrix reads with the Hermitian positive definite basic power matrices $G_i = S_i^{\dagger} S_i$

$$\bar{G} = < S^{\dagger}(t) S(t) > = \sum_{i=1}^{4} \lambda_i S_i^{\dagger} S_i = \sum_{i=1}^{4} \lambda_i G_i .$$

Note that $\mathrm{trace}\, G_i = 1$ for all i and, hence, $\mathrm{trace}\, \bar{G} = \sum_{i=1}^{4} \lambda_i$.

For a symmetric Sinclair matrix the covariance matrix has rank 3, implying that the smallest eigenvalue $\lambda_4 = 0$. Due to this property the 4×4 covariance matrix in the literature is often replaced by a 3×3 covariance matrix right from the beginning.

For the Kennaugh matrix follows

$$S \otimes S^* = \sum_{i=1}^{4} \lambda_i (S_i \otimes S_i^*) \quad \rightarrow \quad K_e = \sum_{i=1}^{4} \lambda_i K_{e,i}$$

i.e., the Kennaugh matrix of a random target is the sum of four
(for backscattering three) elementary Kennaugh matrices of point targets with weights λ_i that are the nonnegative eigenvalues of the corresponding 4×4 (3×3) covariance matrix.

3. References.

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