A Simple Modal Equation for a Single-Wire Line on a Boundary

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1. Introduction

The determination of the propagation constant for a straight wire of infinite length located in a stratified medium is a very fundamental problem in electromagnetic theory. This structure is noticed recently in the problem of the unintentional wave propagation in the ElectroMagnetic Compatibility (EMC). J.R.Wait [1] presented a rigorous formulation of the modal equation for the propagation constant of the transmission current propagation on the wire, but solving the equation is a formidable task. More recent works have dealt with the multilayered media [2].

In this paper, we consider an infinitely long wire on a boundary of two half-spaces of homogeneous dielectric media. We derive generalized distributed constants containing integrals for the traveling mode current along the wire so that the propagation system exhibits the characteristics of a traditional transmission line. Solving these integrals analytically, we obtain a simple modal equation for the propagation constant.

2. Formulation

We consider an infinitely long thin wire of radius a located on a plane interface between two half-spaces of the permittivities, ε_1 and ε_2, respectively, shown in Fig.1. And the permeabilities are assumed to be the same as that of free space μ_0. We have chosen a Cartesian coordinate system (x, y, z) with the wire running on the x-axis. The interface separating the two media is the plane z = 0. A time dependence of the form $e^{j\omega t}$ is assumed.

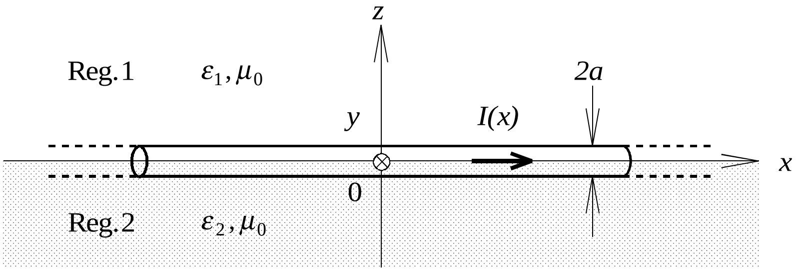

Fig.1 Geometry of a wire on a boundary.

Based on electromagnetic theory, we can express the electric field $\boldsymbol{E}$ by the scalar potential ϕ and the vector potential $\boldsymbol{A}$ at an arbitrary point as follows

$$\boldsymbol{E} = -\nabla\phi - j\omega \boldsymbol{A} \tag{1}$$

where the potentials are expressed by the Hertz vector $\boldsymbol{\Pi}$ as follows

$$\boldsymbol{A} = j\omega\varepsilon\mu\, \boldsymbol{\Pi} \tag{2}$$

$$\phi = -\nabla \cdot \boldsymbol{\Pi} \tag{3}$$

First, the radius of the wire is assumed sufficiently small compared with the wavelength. Hence we can approximate the current flows along the x-direction, and let $\Pi_y = 0$,

$$\boldsymbol{\Pi} = \boldsymbol{j}_x \Pi_x + \boldsymbol{j}_z \Pi_z \tag{4}$$

$$\nabla \cdot \boldsymbol{\Pi} = \frac{\partial}{\partial x}\Pi_x + \frac{\partial}{\partial z}\Pi_z \tag{5}$$

On the wire we obtain the following equation from(1) since its tangential electric field is zero ($\boldsymbol{j}_x \cdot \boldsymbol{E} = 0$).

$$\boldsymbol{j}_x \cdot \nabla\phi = k^2 \boldsymbol{j}_x \cdot \boldsymbol{\Pi} = k^2 \Pi_x \qquad (k^2 = \omega^2 \varepsilon\mu) \tag{6}$$

where $\boldsymbol{j}_x$ is the unit tangential vector along the wire. If the current at the coordinate x along the wire is assumed as $I(x)$ ($-\infty \le x \le \infty$), we can write

$$\frac{\partial\phi}{\partial x} = k^2 \int_{-\infty}^{\infty} I(x') G_x dx' \tag{7}$$

where G_x is the Green function for Π_x. Also, substituting (5) into (3), we can write

$$\phi = -\int_{-\infty}^{\infty} I(x') \frac{\partial G_c}{\partial x} dx' = -\int_{-\infty}^{\infty} \frac{\partial I(x')}{\partial x'} G_c dx' \tag{8}$$

where G_c is expressed by G_x and G_z which is the Green function for Π_z as follows

$$G_c = G_x + \int^x \frac{\partial G_z}{\partial z} dx \tag{9}$$

Next from (7) and (8), we can reduce them generalized transmission line equations as in the following forms by the traveling wave mode method [3].

$$-\frac{\partial\phi(x)}{\partial x} = \zeta(x) I(x), \qquad -\frac{\partial I(x)}{\partial x} = \eta(x)\phi(x) \tag{10}$$

where

$$\zeta(x) = -k_1^2 \int_{-\infty}^{\infty} \frac{I(x')}{I(x)} G_x dx', \qquad \frac{1}{\eta(x)} = \int_{-\infty}^{\infty} \frac{\partial I(x')/\partial x'}{\partial I(x)/\partial x} G_c dx' \tag{11}$$

Following Wait's treatment [1], the Green functions G_x and G_c can be obtained as following integrals.

$$G_x = -\frac{1}{2\pi\omega\varepsilon_1} \int_0^{\infty} \frac{1}{\gamma_1 + \gamma_2} \cdot J_0(\lambda\rho)\lambda\, d\lambda \tag{12}$$

$$G_c = G_x - \frac{1}{2\pi\omega\varepsilon_1} \int_0^{\infty} \frac{\varepsilon_1 - \varepsilon_2}{(\gamma_1 + \gamma_2)(\varepsilon_2\gamma_1 + \varepsilon_1\gamma_2)} \cdot J_0(\lambda\rho)\gamma_1\lambda\, d\lambda \tag{13}$$

where

$$\rho = \sqrt{(x - x')^2 + a^2}, \qquad \gamma_i = \sqrt{k_i^2 - \lambda^2}, \qquad \mathrm{Im}\gamma_i \le 0 \qquad (i = 1,2) \tag{14}$$

Equations (10) are generalized telegrapher's equations for the wire in the transmission line theory. The symbols $\zeta(x)$ and $\eta(x)$ are equivalent to the series impedance and the shunt admittance per unit length of the wire. Namely, the integrodifferential equations which are derived by the electromagnetic theory are reduced formally to the telegrapher's equations.

Their primary parameters $\zeta(x)$ and $\eta(x)$ are represented as integrals involving the unknown current distribution. The current $I(x)$ may be expressed as the sum of two traveling waves.

$$I(x) = I_f e^{-j\beta_0 x} + I_b e^{j\beta_0 x} \tag{15}$$

where β_0 is an assumed propagation constant, and the amplitudes I_f and I_b are arbitrary constants. Because of the infinitely long wire, $\zeta(x)$ and $\eta(x)$ in (11) are not depend on the position of x and the direction of the traveling waves, so that they can be obtained easily as constants ζ_0 and η_0 by letting $x = I_b = 0$ as follows

$$\zeta_0 = -k_1{}^2 \int_{-\infty}^{\infty} e^{-j\beta_0 x'} G_x dx', \qquad \frac{1}{\eta_0} = \int_{-\infty}^{\infty} e^{-j\beta_0 x'} G_c dx' \tag{16}$$

We normalize ζ_0 and η_0 by $j\omega\mu_0$ and $j\omega\varepsilon_1$ of characteristics of region 1. Thus,

$$\zeta_0 = j\omega\mu_0\Psi, \qquad \frac{1}{\eta_0} = \frac{1}{j\omega\varepsilon_1}(\Psi + \Phi) \tag{17}$$

The integrals Ψ and Φ can be solved by assuming $k_i a \ll 1$ (i =1,2) as follows

$$\Psi = \frac{1}{\pi j}\int_{\beta_0}^{\infty} \frac{\lambda}{\gamma_1+\gamma_2}\cdot\frac{\cos(\sqrt{\lambda^2-\beta_0{}^2}\,a)}{\sqrt{\lambda^2-\beta_0{}^2}}d\lambda \cong \frac{1}{2\pi}\left[-j\frac{\pi}{2}+\ln\frac{2}{c_2}-\gamma+\frac{1}{2}+\frac{c_1^2}{c_2^2-c_1^2}\ln\frac{c_1}{c_2}\right] \quad (18)$$

$$\Phi = \frac{1}{\pi j}\int_{\beta_0}^{\infty} \frac{(\varepsilon_1-\varepsilon_2)\gamma_1\lambda}{(\gamma_1+\gamma_2)(\varepsilon_2\gamma_1+\varepsilon_1\gamma_2)}\cdot\frac{\cos(\sqrt{\lambda^2-\beta_0{}^2}\,a)}{\sqrt{\lambda^2-\beta_0{}^2}}d\lambda$$

$$\cong \frac{1}{\pi}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}\left[-j\frac{\pi}{2}+\ln\frac{2}{c_2}-\gamma-\frac{\varepsilon_1}{\varepsilon_1-\varepsilon_2}\left\{\ln\frac{c_2}{c_1}-\frac{\sqrt{c_1^2+d^2}}{d}\ln\frac{c_2(\sqrt{c_1^2+d^2}+d)}{c_1(\sqrt{c_1^2+d^2}+d)}\right\}\right]-\Psi \quad (19)$$

where

$$c_i{}^2 = (k_i{}^2-\beta_0{}^2)a^2, \qquad d^2 = \frac{\varepsilon_2^2 c_1^2-\varepsilon_1^2 c_2^2}{\varepsilon_1^2-\varepsilon_2^2}, \qquad \gamma = 0.5772 \qquad (i=1,2) \quad (20)$$

Finally, the propagation constant β_0 of the current can be obtained from ζ_0 and η_0 .

$$\beta_0 = \sqrt{\zeta_0\psi_0} = \frac{k_1}{\sqrt{1+\Phi/\Psi}} \quad (21)$$

For convenience, we represent an equivalent relative permittivity ε_{re} by the square of β_0 normalized by the free space propagation constant $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$.

$$\varepsilon_{re} \equiv \left(\frac{\beta_0}{k_0}\right)^2 = \frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_0}\cdot\frac{-j\frac{\pi}{2}+\ln\frac{2}{c_2}-\gamma+\frac{1}{2}+\frac{c_1^2}{c_2^2-c_1^2}\ln\frac{c_1}{c_2}}{-j\frac{\pi}{2}+\ln\frac{2}{c_2}-\gamma-\frac{\varepsilon_2}{\varepsilon_2-\varepsilon_1}\left\{\ln\frac{c_1}{c_2}-\frac{\sqrt{c_1^2+d^2}}{d}\ln\frac{c_1(\sqrt{c_2^2+d^2}+d)}{c_2(\sqrt{c_2^2+d^2}+d)}\right\}} \quad (22)$$

3.Numerical result

Equation(22) is an iterational formula of ε_{re} because c_1, c_2 and d are the functions of β_0 as shown in (20). Fig.2 shows the n-iterated equivalent relative permittivity $\varepsilon_{re}^{(n)}$ depending on the initial one $\varepsilon_{re}^{(0)}$. From this result, as the initial value of $\varepsilon_{re}^{(0)}$ on the right of (22), $(\varepsilon_1+\varepsilon_2)/2\varepsilon_0$ would be the best selection, and be used henceforth.

Fig.3 shows the convergence of ε_{re} of the iteration. It is found that ε_{re} converges rapidly for a few iterations. Numerical results are shown in Fig.4 for $\varepsilon_{r1}=1$, $\varepsilon_{r2}=2{,}4{,}8$ where $\varepsilon_{ri}=\varepsilon_i/\varepsilon_0$, and in Fig.5 for $\varepsilon_{r1}=1$, $\varepsilon_{r2}-4=0, -j2, -j4$. From Fig.4, ε_{re} increases as a/λ increases, and is complex though the two media are not dissipative. This means that surface wave is generated in the medium of the lower permittivity, and leaky wave in the medium of the higher one. From Fig.5, the real part of ε_{re} becomes independent on a/λ as the imaginary part of ε_{r2} increases.

4.Conclusion

We derived the simple modal equation for a single-wire line on a boundary by using the traveling wave mode method. Numerical results show that the square of the normalized propagation constant, i.e., the equivalent relative permittivity is larger than the arithmetic mean of permittivities of the two media and has an imaginary part, which means the loss of radiation. It is noted that the traveling wave of the single-wire line on the boundary has compound characteristics of surface wave and leaky wave.

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Reference

[1] James R.Wait, Theory of wave propagation along a thin wire parallel to an interface, Radio Science, Vol.7, No.6, pp.675-679, Jun., 1972.

[2] Renshou Dai, Charles T.Young, Theory and sensing applications of a long antenna on layered media, Radio Science, Vol.32, No.1, pp.43-57, Jan.-Feb., 1997.

[3] Takashi NAKAMURA, Keisuke ODA and Senji YOKOKAWA, Equivalent Transmission Lines of Wire Antenna and the Radiation Mechanism, IEICE, Vol.J71-B, No.9, pp.1028-1036, 1988.

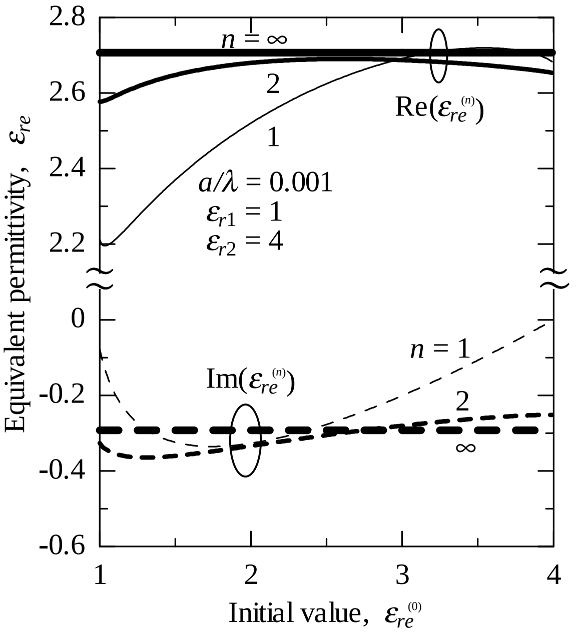

Fig.2 Equivalent permittivity vs. initial value $\varepsilon_{re}^{(0)}$.

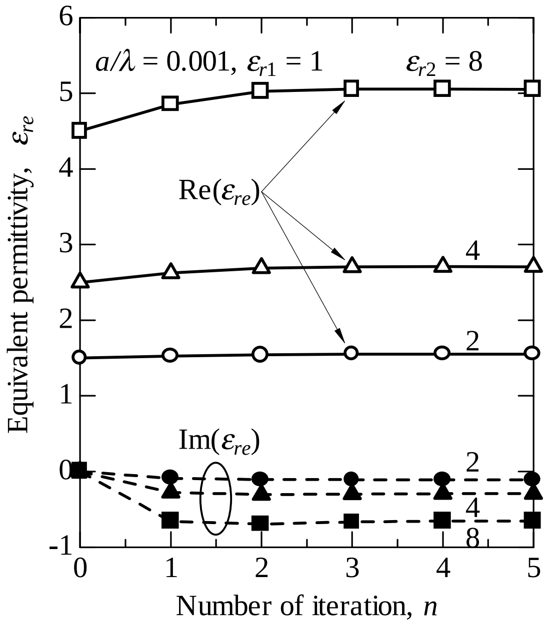

Fig.3 Convergence performance.

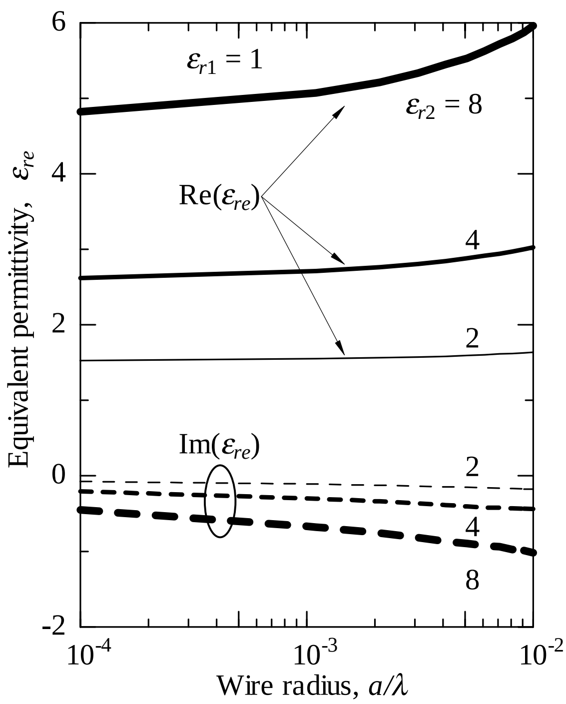

Fig.4 Equivalent permittivity vs. radius in the case of pure dielectric media.

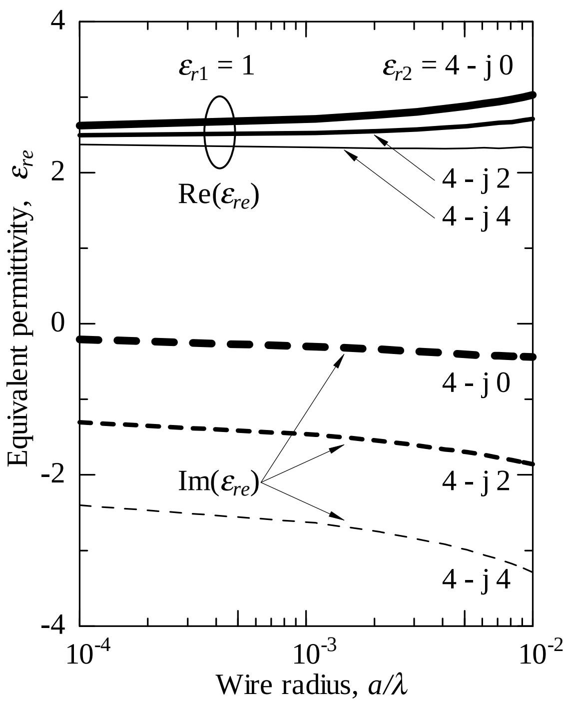

Fig.5 Equivalent permittivity vs. radius in the case of dissipative media.