A GEOMETRIC PERTURBATION THEORY FOR THE ANALYSIS OF PROPAGATING HIGH-FREQUENCY CURRENTS

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1. Introduction

In this paper we provide the theoretical framework to analyze the propagation of high-frequency electric currents along extended and arbitrarily shaped conductors. Practical examples for such conductors are given by broadband traveling wave antennas [1] or nonuniform transmission line structures [2]. They typically have in common to possess smooth boundaries and geometric shapes that can be described by two-dimensional Riemannian surfaces. We use a geometric perturbation expansion to directly link the geometry of these surfaces to the dynamical features of propagating electric currents on these surfaces. As it will turn out these features include as a first order curvature effect the *damping of an electric current due to radiation*. Also the mathematical origin of *continuous reflections along a nontrivial geometry* and the *generation of higher modes* is recognized. However, both phenomena turn out to be second order curvature effects that are present if the first derivative of the curvature does not vanish. This analysis of the electric current provides, of course, analogous conclusions for the accompanying electromagnetic field and therefore should prove to be useful in the context of antenna design, for example.

The basis of our considerations is outlined in Sec. 2 where a magnetic field integral equation for the electric surface current is perturbatively solved in the frequency domain. The key observation is that the single terms of the corresponding Neumann series can be further expanded in terms of the principal curvatures of the conducting surface. This geometric expansion is valid at high frequencies where the wavelength becomes comparable to or smaller than the characteristic geometric dimensions. To become more specific we then assume, in Sec. 3, that the conducting surface has a line-like structure such that a thin-wire approximation can be performed. The resulting formulas are elementary enough to analytically study in Sec. 4 the relation between a propagating electric current and the curvature of the conductor.

2. Scattering expansion and curvature expansion

To begin with we concentrate on a conductor with a two-dimensional smooth surface that resides in a homogeneous medium (like vacuum, for example). We assume the conductor to be a good one such that an incident electromagnetic field induces a surface current $\boldsymbol{k}$, i.e., a current line density, and thus does not significantly penetrate into the conductor. In terms of the magnetic field the general solution of Maxwell's equation in Lorentz gauge reads [3]

$$\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{\nabla}_r \times \int G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')\, d^2\sigma' , \qquad G(\boldsymbol{r},\boldsymbol{r}') = \frac{\exp(jk|\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} , \tag{1}$$

with $\boldsymbol{J}$ the electric current density which appears in Maxwell's equations and $G(\boldsymbol{r},\boldsymbol{r}')$ the retarded Green's function of free space. The time dependence is assumed to be of the form

$\exp(-j\omega t)$. On the surface the coupling between the electromagnetic field and the electric surface current is expressed by means of the boundary condition

$$\boldsymbol{n} \times \boldsymbol{H} = \boldsymbol{k} \quad \text{(on the surface)} \tag{2}$$

where $\boldsymbol{n}$ denotes an outwards pointing normal unit vector. The current $\boldsymbol{k}$ is the sum of an (initially known) source current $\boldsymbol{k}_s$ and an (initially unknown) induced current $\boldsymbol{k}_c$ which is due to scattered electromagnetic fields, $\boldsymbol{k} = \boldsymbol{k}_s + \boldsymbol{k}_c$. In the same way we have the magnetic field as a sum of a primary field $\boldsymbol{H}_s$, which is due to $\boldsymbol{k}_s$, and an induced magnetic field $\boldsymbol{H}_c$, which is due to $\boldsymbol{k}_c$, that is, $\boldsymbol{H} = \boldsymbol{H}_s + \boldsymbol{H}_c$. From this it is straightforward to obtain via Green's theorem an integral equation for the unknown induced current $\boldsymbol{k}_c$ [4],

$$\boldsymbol{k}_c(\boldsymbol{r}) = 2\boldsymbol{n}_r \times \boldsymbol{H}_s^{\text{inc}}(\boldsymbol{r}) + 2\int \boldsymbol{n}_r \times \Big[\boldsymbol{\nabla}_r \times \Big(G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{k}_c(\boldsymbol{r}')\Big)\Big]\, d^2\sigma'\,, \tag{3}$$

with $\boldsymbol{H}_s^{\text{inc}}(\boldsymbol{r})$ the incoming magnetic field which is due to $\boldsymbol{k}_s(\boldsymbol{r}')$ for $\boldsymbol{r} \neq \boldsymbol{r}'$. This integral equation is of the second kind and can be solved by iteration. For this purpose we define the first order current $\boldsymbol{k}_{1c} := 2\boldsymbol{n} \times \boldsymbol{H}_s^{\text{inc}}$ and the functional $F \ldots := 2\int \boldsymbol{n}_r \times [\boldsymbol{\nabla}_r \times (G(\boldsymbol{r},\boldsymbol{r}') \ldots (\boldsymbol{r}'))]\, d^2\sigma'$. This yields the Neumann series

$$\boldsymbol{k}_c = \boldsymbol{k}_{1c} + F\boldsymbol{k}_{1c} + F^2\boldsymbol{k}_{1c} + \ldots = \boldsymbol{k}_{1c} + \boldsymbol{k}_{2c} + \boldsymbol{k}_{3c} + \ldots \tag{4}$$

for the unknown current $\boldsymbol{k}_c$. It has the physical interpretation of a *scattering expansion* where the nth term $\boldsymbol{k}_{nc}$ accounts for contributions of electromagnetic fields which got scattered $(n-1)$-times at the conducting structure.

With the determination of the induced current $\boldsymbol{k}_c$ the dynamical problem is, in principle, completely solved. However, the remaining practical problem consists in evaluating expressions of the form $\boldsymbol{k}_{1c} = F\boldsymbol{k}_s$ and $\boldsymbol{k}_{(n+1)c} = F\boldsymbol{k}_{nc}$ for $n \geq 1$. For this purpose it turns out to be advantageous to simplify the vectorial expressions within the functional F by means of geometric scalar quantities. Since the Green's function significantly contributes to F only at short distances it is reasonable to expand $(F\boldsymbol{k}_s)(\boldsymbol{r})$, $(F\boldsymbol{k}_{nc})(\boldsymbol{r})$ around $\boldsymbol{r}$ in the following way [5, 6]: We first establish a local coordinate frame with $\boldsymbol{r}$ as its origin, i.e., $\boldsymbol{r} = (0,0,0)$, and take as x- and y- axis the principal axes of the surface at $\boldsymbol{r}$ with principal curvatures κ_1 and κ_2, respectively. Their orientation is chosen such that we obtain a right-handed coordinate system at $\boldsymbol{r}$ if the outwards pointing normal vector $\boldsymbol{n}_r$ is chosen as z-axis. Then a Taylor expansion of the third component of $\boldsymbol{r}' = (x', y', z')$ yields

$$z'(x', y') = \frac{1}{2}\Big(\kappa_1(\boldsymbol{r}')x'^2 + \kappa_2(\boldsymbol{r}')y'^2\Big) + \ldots\,. \tag{5}$$

The dots indicate terms of third and higher order in the distance $|\boldsymbol{r} - \boldsymbol{r}'| = |\boldsymbol{r}'| = r'$ and also contain derivatives of the principal curvatures. It is important to note that the scale of r' must be seen in relation to the principal curvatures, i.e., the expansion is a reasonable one for $\kappa_1 r' \ll 1$ and $\kappa_2 r' \ll 1$. This limits the curvature expansion to electrically large regions. It is now possible to show [5] that

$$(F\boldsymbol{k}_s)(\boldsymbol{r}) = \boldsymbol{k}_{1c}(\boldsymbol{r}) = 2\int \frac{G'(\boldsymbol{r},\boldsymbol{r}')}{r'} k_s(\boldsymbol{r}')\Big(\kappa_1(\boldsymbol{r})\boldsymbol{v}^{\theta_n}(\boldsymbol{r}') + \kappa_2(\boldsymbol{r})\boldsymbol{w}^{\theta_n}(\boldsymbol{r}') + \ldots\Big)\, d^2\sigma'\,. \tag{6}$$

Here we defined $G'(\boldsymbol{r},\boldsymbol{r}') := \partial G(|\boldsymbol{r}-\boldsymbol{r}'|)/\partial(|\boldsymbol{r}-\boldsymbol{r}'|)$, $k_s(\boldsymbol{r}') := |\boldsymbol{k}_s(\boldsymbol{r}')|$, and the vectors

$$\boldsymbol{v}^{\theta_n}(\boldsymbol{r}') := \begin{pmatrix} x'^2(\sin\theta_n - \cos\theta_n) \\ x'^2(\sin\theta_n + \cos\theta_n) - 2x'y'(\sin\theta_n - \cos\theta_n) \\ 0 \end{pmatrix}\,, \tag{7}$$

$$\boldsymbol{w}^{\theta_n}(\boldsymbol{r}') := \begin{pmatrix} -y'^2(\sin\theta_n - \cos\theta_n) + 2x'y'(\sin\theta_n + \cos\theta_n) \\ -y'^2(\sin\theta_n + \cos\theta_n) \\ 0 \end{pmatrix}\,, \tag{8}$$

with θ_n the angle between the unit vector $\boldsymbol{e}_{r'} = 1/2(1,1,\kappa_1 x' + \kappa_2 y')$ at $\boldsymbol{r}'$ and the current vector $\boldsymbol{k}_{nc}(\boldsymbol{r}')$. An analogous formula to (6) is valid for $(F\boldsymbol{k}_{nc})(\boldsymbol{r})$. The result (6) accounts for the normally dominant influence on $(F\boldsymbol{k}_{1c})(\boldsymbol{r})$ which is due to the continuously connected neighborhood of the conducting structure around $\boldsymbol{r}$. If the conducting structure consists of several disconnected conductors also possible contributions of these disconnected parts need to be considered. Depending on the specific constellation the curvature expansion can be applied in this case, too.

3. Thin wire approximation

Conducting structures that carry propagating currents, as the afore mentioned antennas or transmission line, are often given by wire structures. If the wires are relatively thin compared to the wavelength considered it is customary to reduce them to one-dimensional lines by integrating out the spatial dimension along the circumference. This is rather trivial as long as the current distribution is assumed to be constant along the circumference. In the high frequency limit it is expected that this assumption does not yield the correct physical picture such that diffraction theory needs to be applied in order to properly take into account the influence of shadow regions and creeping waves [7]. However, if the assumption of a wire radius that is considerably smaller than the wavelength is valid it is possible to apply a thin wire approximation to (6) with the principal curvatures κ_1, κ_2 locally corresponding to the geometry of a section of a torus. This yields the *scalar* relation

$$k_{1c}(\boldsymbol{r}) = 2\rho\kappa(\boldsymbol{r}) \int \frac{G'(\boldsymbol{r},\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} k_s(\boldsymbol{r}')(y-y')^2\, dy' \tag{9}$$

with ρ the wire radius and $\kappa(\boldsymbol{r})$ the curvature of the wire at $\boldsymbol{r}$. Here the coordinate y parameterizes the (one-dimensional) wire such that the corresponding tangent vector $\partial/\partial y$ is normalized to unity. Therefore the expression $|y-y'|$ is a measure for the length of the wire between the points $\boldsymbol{r}$ and $\boldsymbol{r}'$. This length should not be confused with the distance $|\boldsymbol{r}-\boldsymbol{r}'|$. Both expressions only coincide if the wire section between $\boldsymbol{r}$ and $\boldsymbol{r}'$ is straight. More explicitly, we have to lowest order in the curvature

$$|\boldsymbol{r}-\boldsymbol{r}'| = |y-y'|\Big(1 - \frac{\kappa^2(y)}{24}|y-y'|^2 + \dots\Big). \tag{10}$$

The result (9) takes into account the current-current selfinteraction of a single wire. An example for a disconnected structure is a transmission line which consists of a wire that is held at a fixed height $h/2$ above a perfectly conducting ground plane. In this case the relation (9) generalizes to

$$k_{1c}(\boldsymbol{r}) = 2\rho\kappa(\boldsymbol{r}) \int \Big(\frac{G'(\boldsymbol{r},\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} - \frac{1}{\sqrt{1+(\frac{h}{|y-y'|})^2}}\frac{G_h'(\boldsymbol{r},\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\Big) k_s(\boldsymbol{r}')(y-y')^2\, dy' \tag{11}$$

with the modified Green's function $G_h(\boldsymbol{r},\boldsymbol{r}') := \exp(jk\sqrt{|\boldsymbol{r}-\boldsymbol{r}'|^2+h^2})/\sqrt{|\boldsymbol{r}-\boldsymbol{r}'|^2+h^2}$.

4. First and second order evaluation: Radiation effects, continuous reflections and generation of higher modes

To gain some physical insight we consider a traveling wave current of the form $\boldsymbol{k}_s = \boldsymbol{k}_0 \exp(jky)$, i.e., the wave travels towards increasing y. It can be plugged into the relations (9) and (11). To evaluate the resulting integrals we first express $|\boldsymbol{r}-\boldsymbol{r}'|$ in terms of $|y-y'|$ according to (10). It is then interesting to note that if either the lowest order approximation $|\boldsymbol{r}-\boldsymbol{r}'| \approx |y-y'|$ is used or the curvature is constant the integrands depend only on the difference $|y-y'|$ except for the factor $\boldsymbol{k}_s(y') = \boldsymbol{k}_0 \exp(jky')$. Then the integration variable can be shifted according to $y' \longrightarrow y' + y$ and it is possible to eliminate the y dependence from the integrals by pulling a factor $\exp(jky)$ in front of them. This yields a first order current of the form

$$k_{1c}(y) = \frac{\rho\kappa k_0}{2\pi} C(k) \exp(jky) \tag{12}$$

with a complex factor $C(k)$ which depends only on k (or possibly geometric parameters like h in (11)). The explicit integrals contained in the factor $C(k)$ are rather elementary and can, after a regularization procedure which subtracts out a quasistatic diverging contribution, be either analytically estimated or numerically evaluated. It is then seen that $C(k)$ is dominated by a negative real part. In view of (12) this means that the traveling wave current experiences a *damping* while passing along a *curved* wire section. This damping is attributed to radiation and can be expressed by means of the notion of radiation resistance. However, the forward traveling wave remains to be a forward traveling wave of the same frequency.

We now assume that the curvature along the wire region considered is *not* constant and the full expression (10) is used in the replacement of $|\boldsymbol{r} - \boldsymbol{r}'|$ within the integrals. In this case it is not possible to immediately eliminate the y-dependence from the integrand by pulling out a factor $\exp(jky)$. Rather it is required to analyze the spectrum of the resulting expressions (e.g. by Fourier analysis) which depend on the *change of curvature*. For example, it is possible to expand $\kappa(y)$ itself in a Taylor series, making the dependence on the derivative $\kappa'(y)$ more explicit. Due to the minus sign in (10) in front of the curvature it is then recognized that the spectrum will not only contain modes with a wavenumber *different* from k but also backwards traveling waves which are interpreted as *reflections* of the forward traveling wave while passing along wire sections of *variable* curvature.

In conclusion we have thus recovered the main dynamical features of a propagating electric current and analytically shown their connection to the geometry of the underlying conductor. Our current research concentrates on the extension of these results to geometric regions that are electrically short like regions close to sharp bends or edges.

References

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