Full Wave Analysis of Dispersion Properties for a Rectangular Waveguide Grating

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Abstract – Dispersion characteristics of a rectangular waveguide grating for microwave amplifier applications are studied in a rigorous mathematical manner on the basis of the singular integral equation method. Over a wide range of grating parameters, numerical results are in a surprisingly good agreement with those obtained before by a simplified consideration. Meanwhile, the difference between two approaches becomes significant with the decrease in the lamellar thickness.

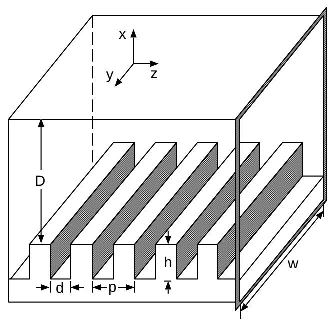

Figure 1: Geometry of the problem.

I Introduction

Metallic periodic grating structures have numerous applications in the microwave and millimeter wave techniques. Particularly, they serve as slow wave structures in a lot of microwave electron devices [1-3]. Very often theoretical considerations of such structures are based on simplified treatments of the boundary value problems associated with the periodic gratings. In the case of rectangular grooves (lamellar gratings), the fields inside them can be expressed in terms of infinite Fourier series. However, for the sake of simplicity, only the first term of the series is often taken into account [1-2],[4-5] (later called single mode approximation: SMA). Results were in a good agreement with the experimental observations [6] when the period of the structure is much less than the vacuum wavelength.

Meanwhile, due to such widely present and probably potential applications of periodic structures with lamellar gratings, the development of a more universal and flexible mathematical description is most desirable. With such a useful technique, one could more thoroughly realize the validity of the above mentioned simplified treatment and reveal the situations where it can not provide a proper precision.

In this paper, we develop a new approach based on the singular integral equation (SIE) method [7-8], in order to carry out the full wave modal analysis of a rectangular waveguide grating rigorously. Numerical results are compared with those obtained by the SMA method, which is presently available for a specific hybrid (TE_y) mode. A brief illustration of dispersion characteristics for a more natural (TM) mode calculated by our method is also shown in this paper.

II Boundary Value Problem and Initial Singular Integral Equation

A rectangular waveguide illustrated in Fig.1 is considered. According to [4-5], such a structure supports TE_{ymn} mode, where $E_y = 0$. For this polarization, all electromagnetic field components can be expressed through $H_y = H_{ym}(x,z)\sin(\frac{\pi m}{w}y)e^{-i\omega t}$. For $H_{ym}(x,z) \equiv \Psi(x,z)$, the Helmholtz equation reads:

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \nu_y^2)\ \Psi(x,z) = 0 \qquad (1)$$

where $\nu_y^2 = k^2 - k_{ym}^2$, $k = \frac{\omega}{c}$, $k_{ym} = \frac{\pi m}{w}$, with Neumann's boundary conditions on the top and bottom walls:

$$\frac{\partial \Psi(x,z)}{\partial \mathbf{n}} = 0 \qquad (2)$$

where $\mathbf{n}$ is the unit normal to the surface.

From the Floquet theorem, we can restrict the consideration to only one grating period, $0 < z < p$. Over the grating ($0 < x < D$), $\Psi(x,z)$ can be found in the form of spatial harmonic series:

$$\Psi(x,z) = \sum_{n=-\infty}^{\infty} a_n \cosh[\chi_n(D-x)]\ e^{ik_n z} \quad (3)$$

where $k_n = k_z + nk_0$, $\chi_n = \sqrt{k_n^2 - \nu_y^2}$, $k_0 = 2\pi/p$, and a_n $(n = 0, \pm 1, \pm 2 \cdots)$ are unknown amplitudes of spatial harmonics.

In the groove ($-h < x < 0$, $0 < z < d$), the solution can be expressed in terms of Fourier series:

$$\Psi(x,z) = \sum_{n=0}^{\infty} X_n \cosh[\nu_n(x+h)]\ \cos\xi_n z \quad (4)$$

where $\xi_n = \frac{\pi n}{d}$, $\nu_n = \sqrt{\xi_n^2 - \nu_y^2}$ and X_n are unknown modal coefficients.

The next step is to match the electromagnetic fields at the interface between the two regions, where the continuity of $\Psi(x,z)$ and $\frac{\partial \Psi(x,z)}{\partial x}$ at $x = 0$, $0 < z < d$ is required. Introducing the new function $\frac{\partial \Psi}{\partial x}(x,z)|_{x=0} = F(z)$, $0 < z < d$, we can express a_n in terms of $F(z)$ by inverting the differentiation of (3) at $x = 0$. Substituting a_n into (3), we obtain the following integral representation for $\Psi(x,z)|_{x\to+0} \equiv \Psi^+(z)$:

$$\Psi^+(z) = -\int_0^d G_0(z-z')\ F(z')\ dz', \ \ 0<z<d \quad (5)$$

where

$$G_0(x) = \frac{1}{p}\sum_{n=-\infty}^{\infty} \frac{\mathrm{ctanh}(\chi_n D)}{\chi_n}\ e^{ik_n x}.$$

Proceeding in the same way with the field inside the groove and taking into account the continuity of $\frac{\partial \Psi}{\partial x}(x,z)$ at $x = 0$, we have $\Psi(x,z)|_{x\to-0} \equiv \Psi^-(z)$:

$$\Psi^-(z) = \int_0^d G_1(z,z')\ F(z')\ dz', \ \ 0<z<d \quad (6)$$

where

$$G_1(z,z') = \frac{2}{d}\sum_{n=0}^{\infty} \epsilon_n \frac{\mathrm{ctanh}(\nu_n h)}{\nu_n} \cos\xi_n z \cos\xi_n z',$$

$$\epsilon_n = \begin{cases} 1/2, & n = 0 \\ 1, & n > 0 \end{cases}.$$

Thus, for the the continuity of $\Psi(x,z)\Big|_{x=0}$:

$$\Psi^+(z) = \Psi^-(z), \quad 0<z<d \qquad (7)$$

we finally obtain the integral equation:

$$\int_0^d G(z,z')\ F(z')\ dz' = 0, \ \ 0<z<d \qquad (8)$$

where

$$G(z,z') = G_0(z-z') + G_2(z-z') + G_2(z+z'),$$

$$G_2(x) = \sum_{n=0}^{\infty} \epsilon_n \frac{\mathrm{ctanh}(\nu_n h)}{\nu_n} \cos\xi_n x.$$

The integral equation (8) is an exact dispersion relation, corresponding to the initial boundary value problem (1) with (2). It contains complete information about dispersion properties of the structure, where all spatial harmonics of the field over the groove and all Fourier harmonics inside the groove are taken into account. However, the kernel of (8) has a logarithmic weak singularity as $z \to z'$ [8]; i.e., $G_0(z,z') \to A\ln|z-z'|$ for $z \to z'$, where A is constant. As is well known, such an integral equation of the first kind is very inconvenient for direct numerical analysis, because the numerical solution is very unstable with respect to small perturbations. In the next section, one specific method for deriving stable numerical results will be presented.

III Description of the Numerical Method

Since the more singular behavior of the kernel as $z \to z'$ provides the more stable solution [9], we differentiate (8) with respect to z and gain the integral equation with a stronger singularity. The resulting equation has a kernel with the Cauchy-type singularity:

$$\int_0^d G'(z,z')\ F(z')\ dz' = 0 \tag{9}$$

where

$$G'(z,z') = \frac{\partial G(z,z')}{\partial z} \sim \frac{1}{z-z'}, \text{ as } z \to z'.$$

Further, the additive constant for the solution of (9) can be defined from an auxiliary condition, which can be obtained by the integration of (8) over the interval $(0,d)$:

$$\int_0^d M(z')\ F(z')\ dz' \ = \ 0 \tag{10}$$

where

$$M(z) = \frac{2}{p}\sum_{n=-\infty}^{\infty} \frac{\text{ctanh}(\chi_n D)}{k_n \chi_n}\ \sin\frac{k_n d}{2}\ e^{ik_n(\frac{d}{2}-z)} + \frac{\text{ctanh}(\nu_0 h)}{\nu_0}.$$

Thus, the integral equation (9) with the auxiliary condition (10) defines a unique solution with a proper behavior near the edges of lamellars $(x=0, z=0,d)$: $F(z) \sim (z(d-z))^{-1/3}$. When $F(z) = \text{const}$, we can easily obtain, from (10), the SMA dispersion relation coincided with that in [4].

For the numerical analysis of the integral equation (9) with the auxiliary condition (10), we use the direct numerical method developed in [10]. It results in the dispersion relation which is appropriate for numerical analysis of rectangular waveguide gratings with any reasonable parameters, including overmoded waveguides. By this approach, the multimodal content of the fields as well as the singularity of the fields near the edges of the lamellars can be taken into account correctly.

IV Numerical Results

For the shallow and deep rectangular grating waveguides, which have been studied in detail theoretically in the scope of SMA [4-5] as well as experimentally [6], we see quite a good agreement between the SIE and SMA methods. The difference between two methods does not exceed 2% for the first four TE_y modes for both deep and shallow gratings (see Fig. 2 for the deep grating). The overall dispersion behavior of the shallow grating calculated from the SIE method is shown in Fig. 3(a).

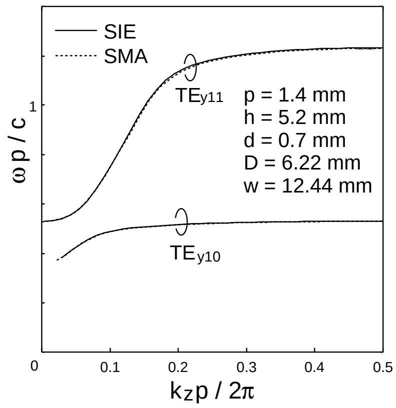

Figure 2: Dispersion curves for the first two modes for the deep grating.

In the SIE approach, singularities near the edges of lamellars are correctly taken into account. Meanwhile, by the SMA approach, we cannot explain the local behavior of the field which is quite different from the average dispersion characteristic of the structure, especially near the edges of lamellars. As a result, differences between the two approaches become more significant when the ratio d/p increases. For a grating with the same values of p, h and w as those in Fig. 3, $D = 3.116$ mm and $d/p = 0.9$, the relative errors of the SMA approach are larger, reaching 5.4%.

In [4-5], the TE_y mode was considered, since it provided simplicity of analysis. However, the TE_y mode is not a natural mode in the rectangular waveguide grating, where any electromagnetic field can be expressed as a linear combination of the two more natural polarizations $TM_{mn}(H_z = 0)$ and $TE_{m'n'}(E_z = 0)$. On the other hand, as for the beam-wave interaction where only the TM mode can take part in the coupling process, the simplified approach does not seem sufficient for the analysis. This is because, in particular, the process of beam bunching is more sensitive to the lo-

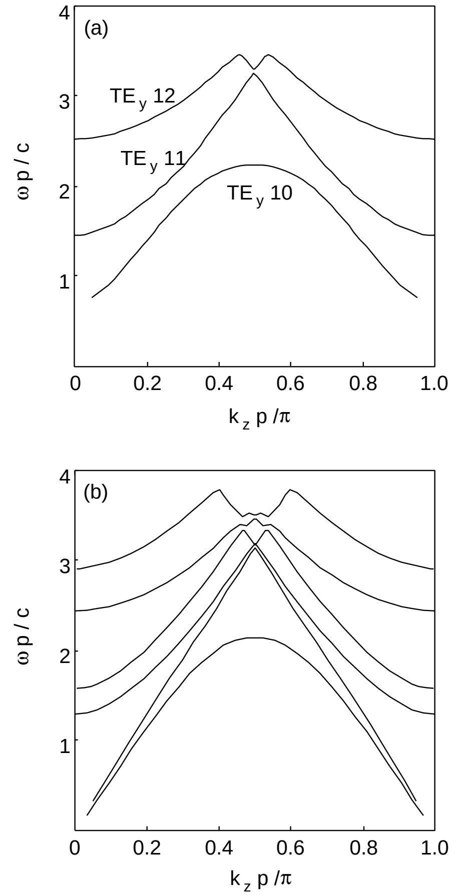

Figure 3: Dispersion curves calculated from SIE for the shallow grating; (a) TE_y, (b) TM polarizations, parameters; $p = 3.556, h = 1.588, d = 1.778, D = 5.08$ and $w = 15.799$ [mm].

cal field behavior than to the average dispersion properties of the cold structure. Therefore, from the above viewpoint, it is reasonable to extend our rigorous approach to the TM mode analysis, with only a little modification in mathematical treatment. Dispersion curves for the TM polarization calculated from the SIE method is presented in Fig. 3(b). Detailed considerations on the behavior of TM modes will be given later.

V Discussion and Conclusion

In this work, full wave modal analysis of dispersion properties for the TE_y mode in the rectangular waveguide grating has been carried out on the basis of the SIE method. However, numerical difficulties came with a logarithmic weak singularity in the kernel, where the solution was rather unstable with respect to small perturbations. The method applied here enabled us to effectively overcome the above mentioned difficulties. Results were compared with those obtained before in the scope of SMA. A surprisingly good agreement between SMA and SIE approaches were demonstrated for both deep and shallow rectangular gratings. However, the difference between two approaches became significant with increasing the ratio d/p, since the field behavior characterized by each approach was much different especially near the edges of lamellars. Similar results were also expected for the case of TM polarization.

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