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THE INVERSE SCATTERING BASED ON THE RECONSTRUCTION OF EXTENDED T-MATRIX ELEMENTS UNMEASURED DIRECTLY FROM SCATTERED WAVES

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1. Introduction

It is worthwhile studying the inverse scattering problem of reconstructing the internal physical properties of an object from knowledge of the scattered microwaves with application to medical diagnosis, underground prospection, and nondestructive examination. In numerous previous studies, the problem is recasted to a nonlinear optimization problem that the scattered wave by the reconstructed object is made to be close to the measured scattered wave, where algorithms such as gradient methods or the genetic algorithm are used and the direct scattering problem is solved repeatedly. In the case that the initial guess of the object is far from the original one, we suffer from traps of local minimum. With the aim of decreasing the difficulty, some researchers dealt with the source type integral equation and investigated the nonradiating equivalent current which contributes nothing to the scattered waves outside the object[1, 2].

The authors have formulated the scattering problem using T-operator which transforms incident waves to the equivalent currents, and have been studying the inverse scattering problem based on the reconstruction of T-operator[3]. In this paper, the T-operator is expressed in a matrix form by using orthonormal basis functions. As a result the matrix elements are separated into two parts; one is directly measured from the scattered waves and the other is not. The matrix with the first part is usually called the T-matrix[4]. We here call the matrix with the two parts the extended T-matrix. An iterative algorithm is proposed where the object and unmeasured elements are reconstructed by decreasing a residual error of the equivalent current in the least square approximation. Numerical examples are also shown.

2. Formulation of the problem

Let us consider a scattering problem of a cylindrical object located in a region $R_{\rm V}$ of free space under E-wave time-harmonic excitations. The geometry is shown in Fig. 1. The time factor $\exp(\mathrm{j}\omega t)$ is suppressed hereafter. The object is described by the object function $\varepsilon_\Delta(\mathbf{r}'') = \varepsilon_{\rm r}(\mathbf{r}'') - 1$, where $\varepsilon_{\rm r}$ is the dielectric constant. We denote the scattered wave by $u_{\rm s}$, which is measured in the region $R_{\rm S}$, the incident wave by $u_{\rm in}$, and the total wave by $u_{\rm t}$. These waves and the object function satisfy the integral equations:

$$u_{\rm s}(\mathbf{r}) = \int_{R_{\rm V}} G(\mathbf{r}, \mathbf{r}'') J_{\rm eq}(\mathbf{r}'') \mathrm{d}\mathbf{r}'', \qquad \mathbf{r} \in R_{\rm S} \tag{1}$$

$$u_{\rm t}(\mathbf{r}') = u_{\rm in}(\mathbf{r}') + \int_{R_{\rm V}} G(\mathbf{r}', \mathbf{r}'') J_{\rm eq}(\mathbf{r}'') \mathrm{d}\mathbf{r}'', \qquad \mathbf{r}' \in R_{\rm V} \tag{2}$$

where G is Green's function given by $G(\mathbf{r}', \mathbf{r}'') = -\frac{\mathrm{j}}{4}\mathrm{H}_0^{(2)}(k|\mathbf{r}' - \mathbf{r}''|)$, k is the wavenumber in free space, $\mathrm{H}_n^{(2)}$ is the Hankel function of the second kind of order n, and $J_{\rm eq}(\mathbf{r}')$ is the equivalent current defined by

$$J_{\rm eq}(\mathbf{r}'') = k^2 \varepsilon_\Delta(\mathbf{r}'') u_{\rm t}(\mathbf{r}'') \tag{3}$$

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Let $R_{\rm V}$ be the circular region of $|\mathbf{r}''| \le b$ and J_m be the Bessel function of order m. When the inner product on $R_{\rm V}$ is defined by $\langle f(\mathbf{r}), g(\mathbf{r})\rangle = \int_{R_{\rm V}} f^*(\mathbf{r})g(\mathbf{r})\mathrm{d}\mathbf{r}$, where the asterisk denotes the complex conjugate, we find that the set of functions with two indices:

$$\eta_m^{m'}(\mathbf{r}'') = \frac{1}{\sqrt{2\pi c_m^{m'}}}\mathrm{J}_m(k_m^{m'}\rho'')\exp(\mathrm{j}m\phi''), \qquad m = 0, \pm1, \pm2, \cdots; \quad m' = 1, 2, \cdots \tag{4}$$

is an orthonormal set on $R_{\rm V}$. Here, $c_m^{m'}$ is defined for normalization as $c_m^{m'} = \int_0^b \mathrm{J}_m^2(k_m^{m'}\rho'')\rho''\mathrm{d}\rho''$ and $k_m^{m'}$ are determined as solutions of the equation

$$\frac{\mathrm{J}_m(k_m^{m'}b)}{k_m^{m'}\mathrm{J}_{m+1}(k_m^{m'}b)} = \frac{\mathrm{J}_m(kb)}{k\mathrm{J}_{m+1}(kb)}, \quad \text{for any } m \tag{5}$$

where $k_m^{m'} < k_m^{m'+1}$ and there is only one solution $k_m^{I(m)}$ satisfying $k_m^{m'} = k$. Once the orthonormal set of functions $\{\eta_m^{m'}(\mathbf{r}'')\}$ is specified, it may be possible to represent a given function which is piecewisely continuous on $R_{\rm V}$ by a linear combination of those functions.

We introduce the T-operator which builds up a relation between the any incident wave and the equivalent current, using

$$J_{\rm eq}(\mathbf{r}'') = \int_{R_{\rm V}} T(\mathbf{r}'', \mathbf{r}')u_{\rm in}(\mathbf{r}')\mathrm{d}\mathbf{r}' \tag{6}$$

Here we expand the T-operator in terms of the orthonormal functions as

$$T(\mathbf{r}'', \mathbf{r}') = \sum_{m,m'}\sum_{n,n'} \eta_m^{m'}(\mathbf{r}'')\left[{}_m^{m'}\mathsf{T}_n^{n'}\right]\eta_n^{*n'}(\mathbf{r}'); \quad \left[{}_m^{m'}\mathsf{T}_n^{n'}\right] = \iint_{R_{\rm V}R_{\rm V}} \eta_m^{*m'}(\mathbf{r}'')T(\mathbf{r}'', \mathbf{r}')\eta_n^{n'}(\mathbf{r}')\mathrm{d}\mathbf{r}''\mathrm{d}\mathbf{r}' \tag{7}$$

We also expand Green's function $G(\mathbf{r}'', \mathbf{r}')$ in the same way and express its coefficient by $\left[{}_m^{m'}\mathsf{G}_n^{n'}\right]$. Because the shape of $R_{\rm V}$ is circular, after some calculations we find $\left[{}_m^{m'}\mathsf{G}_n^{n'}\right] = 0$ if $m \ne n$ and $\left[{}_m^{m'}\mathsf{G}_m^{n'}\right] = 0$ if $m' \ne n'$, $m' \ne I(m)$, and $n' \ne I(m)$.

For the sake of simplicity, let us suppose that incident waves are given by

$$u_{\rm in}(\mathbf{r}'; n) = \eta_n^{I(n)}(\mathbf{r}'), \qquad n = 0, \pm1, \pm2, \cdots, \pm N \tag{8}$$

From Eqs.(6) and (7), the equivalent current by each incident wave are given by

$$J_{\rm eq}(\mathbf{r}''; n) = \sum_{m,m'} \eta_m^{m'}(\mathbf{r}'')\left[{}_m^{m'}\mathsf{T}_n^{I(n)}\right] \tag{9}$$

Green's function is also written by

$$G(\mathbf{r}, \mathbf{r}'') = -\frac{\mathrm{j}}{4}\sum_m \mathrm{H}_m^{(2)}(k\rho)\mathrm{J}_m(k\rho'')\exp(\mathrm{j}m[\phi - \phi'']) \qquad \text{for} \quad |\mathbf{r}| > |\mathbf{r}''| \tag{10}$$

Substituting Eqs.(9) and (10) into Eq.(1) and noting the orthonormality of $\{\eta_m^{m'}(\mathbf{r}'')\}$, we get

$$u_{\rm s}(\mathbf{r}; n) = \sum_m \left(-\frac{\mathrm{j}}{4}\right)\sqrt{2\pi c_m^{I(m)}}\left[{}_m^{I(m)}\mathsf{T}_n^{I(n)}\right]\mathrm{H}_m^{(2)}(k\rho)\exp(\mathrm{j}m\phi) \tag{11}$$

From Eq.(11) we know that $\left[{}_m^{I(m)}\mathsf{T}_n^{I(n)}\right]$ is related to the scattered wave for any incident wave (measured elements). In other words, $\left[{}_m^{m'}\mathsf{T}_n^{I(n)}\right]$ for $m' \ne I(m)$ cannot be observed directly from the scattered waves(unmeasured elements). Accordingly we can divide the equivalent current into two components as

$$J_{\rm eq}(\mathbf{r}''; n) = J_{\rm eq}^{\rm M}(\mathbf{r}''; n) + J_{\rm eq}^{\rm NM}(\mathbf{r}''; n) \tag{12}$$

$$J_{\rm eq}^{\rm M}(\mathbf{r}''; n) = \sum_m \eta_m^{I(m)}(\mathbf{r}'')\left[{}_m^{I(m)}\mathsf{T}_n^{I(n)}\right], \quad J_{\rm eq}^{\rm NM}(\mathbf{r}''; n) = \sum_{m,\ m' \ne I(m)} \eta_m^{m'}(\mathbf{r}'')\left[{}_m^{m'}\mathsf{T}_n^{I(n)}\right] \tag{13}$$

3. Inverse algorithm

Let us introduce the cost functional defined by

$$\Omega(J_{\rm eq}^{\rm NM}(\mathbf{r}'; n), \varepsilon_\Delta(\mathbf{r}')) = \sum_{n=-N}^{N} \left\| J_{\rm eq}(\mathbf{r}'; n) - k^2\varepsilon_\Delta(\mathbf{r}')u_{\rm t}(\mathbf{r}'; n)\right\|^2 \tag{14}$$

where $u_{\rm t}(\mathbf{r}';n)$ is the total wave generated by $J_{\rm eq}(\mathbf{r}'';n)$ in $R_{\rm V}$ and $\|f(\mathbf{r})\|^2 = \langle f(\mathbf{r}), f(\mathbf{r})\rangle$. We can reduce the inverse scattering problem to the minimization of Eq.(14) to find the optimal $\varepsilon_\Delta(\mathbf{r}')$. We introduce a set of orthogonal functions $\{\Phi_l\}$ $(l = 1,\cdots,L)$ over $R_{\rm V}$, expand the object function as $\varepsilon_\Delta(\mathbf{r}'') = \sum_{l=1}^{L} \varepsilon_l \Phi_l(\mathbf{r}'')$ and get

$$\left[{}^{p'}_{p}\mathsf{F}(l)^{m'}_{m}\right] = \int \eta^{*p'}_{p}(\mathbf{r}'')\Phi_l(\mathbf{r}'')\eta^{m'}_{m}(\mathbf{r}'')d\mathbf{r}'', \qquad \left[{}^{p'}_{p}\mathsf{H}(l)^{n'}_{m}\right] = \sum_{m'} \left[{}^{p'}_{p}\mathsf{F}(l)^{m'}_{m}\right]\left[{}^{m'}_{m}\mathsf{G}^{n'}_{m}\right] \tag{15}$$

Here $\left[{}^{p'}_{p}\mathsf{F}(l)^{m'}_{m}\right]$ and $\left[{}^{p'}_{p}\mathsf{H}(l)^{n'}_{m}\right]$ are independent of the object and the equivalent current, and calculated only once in advance at the beginning of simulations.

At first, we suppose that $\varepsilon_\Delta(\mathbf{r}')$ is constant and $J^{NM}_{\rm eq}(\mathbf{r}';n)$ is variable, and rewrite Eq.(14) into

$$\Omega = \sum_{n=-N}^{N} \sum_{p,p'} \left| \sum_{m,\ m'\neq I(m)} \left[{}^{p'}_{p}\mathsf{A}^{m'}_{m}\right]\left[{}^{m'}_{m}\mathsf{T}^{I(n)}_{n}\right] - \left[{}^{p'}_{p}\mathsf{B}^{I(n)}_{n}\right] \right|^2 \to \min \tag{16}$$

with

$$\left[{}^{p'}_{p}\mathsf{A}^{m'}_{m}\right] = \delta_{pm}\delta_{p'm'} - k^2 \sum_{l=1}^{L} \varepsilon_l \left[{}^{p'}_{p}\mathsf{H}(l)^{m'}_{m}\right] \tag{17}$$

$$\left[{}^{p'}_{p}\mathsf{B}^{I(n)}_{n}\right] = k^2 \sum_{l=1}^{L} \varepsilon_l \left[{}^{p'}_{p}\mathsf{F}(l)^{I(n)}_{n}\right] - \sum_{m} \left[{}^{p'}_{p}\mathsf{A}^{I(m)}_{m}\right]\left[{}^{I(m)}_{m}\mathsf{T}^{I(n)}_{n}\right] \tag{18}$$

where δ is the Kronecker delta. Eq.(16) involves unknowns as $\left[{}^{m'}_{m}\mathsf{T}^{I(n)}_{n}\right]$ of $m' \neq I(m)$ and is equivalent to $\sum_{p,p'} \left|\sum_{m,\ m'\neq I(m)} \left[{}^{p'}_{p}\mathsf{A}^{m'}_{m}\right]\left[{}^{m'}_{m}\mathsf{T}^{I(n)}_{n}\right] - \left[{}^{p'}_{p}\mathsf{B}^{I(n)}_{n}\right]\right|^2 \to \min$ for each n.

Next, we suppose that $J^{\rm NM}_{\rm eq}(\mathbf{r}';n)$ is constant and $\varepsilon_\Delta(\mathbf{r}')$ is variable, and rewrite Eq.(14) into

$$\Omega = \sum_{n=-N}^{N} \sum_{p,p'} \left| \left[{}^{p'}_{p}\mathsf{T}^{I(n)}_{n}\right] - \sum_{l=1}^{L} \varepsilon_l \left[{}^{p'}_{p}\mathsf{C}(l)_{n}\right] \right|^2 \to \min \tag{19}$$

with

$$\left[{}^{p'}_{p}\mathsf{C}(l)_{n}\right] = k^2 \left(\left[{}^{p'}_{p}\mathsf{F}(l)^{I(n)}_{n}\right] + \sum_{m,m'} \left[{}^{p'}_{p}\mathsf{H}(l)^{m'}_{m}\right]\left[{}^{m'}_{m}\mathsf{T}^{I(n)}_{n}\right] \right) \tag{20}$$

Equations (16) and (19) are solved as the linear least-squares problem using the QR decomposition. The inverse algorithm is summarized as follows:

Step 1: Set the initial value of $\varepsilon_\Delta(\mathbf{r}')$.
Step 2: Update $J^{NM}_{\rm eq}(\mathbf{r}';n)$ by Eq.(16).
Step 3: Update $\varepsilon_\Delta(\mathbf{r}')$ by Eq.(19) and go back to Step 2 and repeat Steps 2 and 3 until convergence.

4. Numerical Examples

Assuming that the object is axisymmetric, we can find $\left[{}^{p'}_{p}\mathsf{F}(l)^{m'}_{m}\right] = 0$, $\left[{}^{p'}_{p}\mathsf{H}(l)^{m'}_{m}\right] = 0$, $\left[{}^{p'}_{p}\mathsf{A}^{m'}_{m}\right] = 0$, $\left[{}^{p'}_{p}\mathsf{B}^{I(n)}_{n}\right] = 0$, $\left[{}^{p'}_{p}\mathsf{C}(l)_{n}\right] = 0$, and $\left[{}^{p'}_{p}\mathsf{T}^{I(n)}_{n}\right] = 0$ for $p \neq m$ or $p \neq n$.

We will numerically reconstruct two kinds of circular cylinders: models A and B. Model A is a homogeneous cylinder with $\varepsilon_\Delta = 0.44$(the refractive index is 1.2) and the radius is λ, where $\lambda = 2\pi/k$ is the wavelength in free space. Model B is an axisymmetric two-layered cylinder whose inner layer has $\varepsilon_\Delta = (0.8, -0.2)$ and the radius is 0.5λ and whose outer layer has $\varepsilon_\Delta = (0.4, -0.4)$ and the radius is λ. Each model is located at the origin. The $R_{\rm V}$ radius $b = 2\lambda$ is used. Let $\Phi_l(\mathbf{r}'')$ be the pulse function such that $\Phi_l(\mathbf{r}'') = 1$ for $b(l-1)/L < |\mathbf{r}''| < bl/L$; otherwise $\Phi_l(\mathbf{r}'') = 0$ over $l = 1,\cdots,L$, where $L = 32$ is used in this computation. The sums of infinite series with respect to m and p in Eqs.(16) and (18)–(20) are truncated at $m,\ p = -N \sim N$, and also the sums with respect to m' and p' are truncated at $m',\ p' = 1 \sim 32$. Here, $N = 10$ is used in this case.

Figure 2 shows the values of the cost functional Ω and of a residual error of the reconstructed ε_Δ as functions of the number of iterations. Both of the values decrease rapidly within first 50 iterations and seem to almost converge after 100 iterations for both models A and B. Figures 3 and 4 show the reconstructed profiles of both models, respectively. The computation time for 200 iterations was 2 minutes by Compaq W8000 with Xeon 2.4GHz processor.

5. Concluding Remarks

As a result of the extended T-matrix expression in terms of orthonormal basis functions, the measured elements of extended T-matrix, which are directly connected to the scattered waves, can be extracted explicitly. We have proposed an iterative inverse algorithm where the object and the unmeasured elements of extended T-matrix are updated by decreasing the cost functional in the least square approximation. The algorithm avoids employing a nonlinear optimization algorithm and solving the direct scattering problem. Numerical examples show that the algorithm works well for the objects for which the Born approximation becomes completely invalid. The inversion algorithm will work more powerfully if we use a priori information about the object and/or some frequency band.

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References

[1] T. M. Habashy, M. L. Oristaglio, and A. T. de Hoop, *Radio Sci.*, **29**(4), 1101–1118, 1994.

[2] S. Caorsi and G. L. Gragnani, *Radio Sci.*, **34**(1), 1–8, 1999.

[3] K. Ishida, H. Furukawa, and M. Tateiba, In *Proc. 2000 International Symposium on Antennas and Propagation, vol. 3*, 1299–1302, 2000.

[4] P. C. Waterman, *J. Acoust. Soc. Am.*, **45**(6), 1417–1429, 1969.

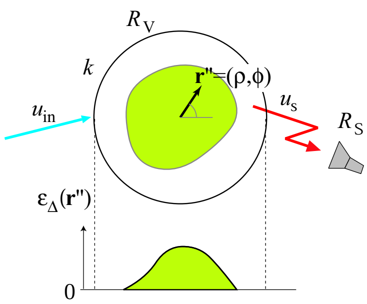

Figure 1: Geometry

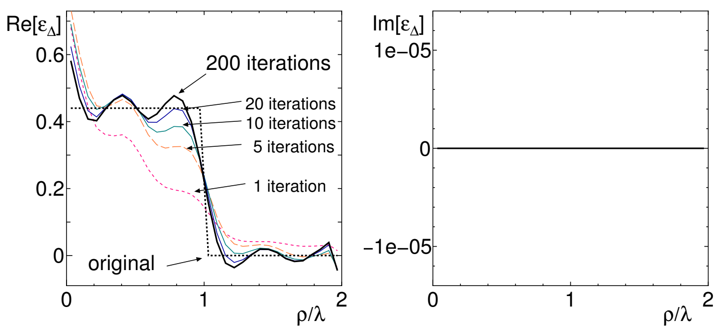

Figure 3: Reconstructed profiles of model A

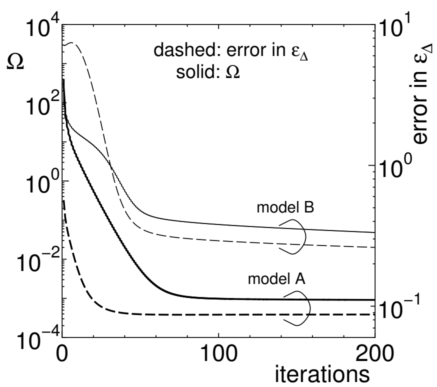

Figure 2: The values of the cost functional and an error of reconstructed ε_Δ

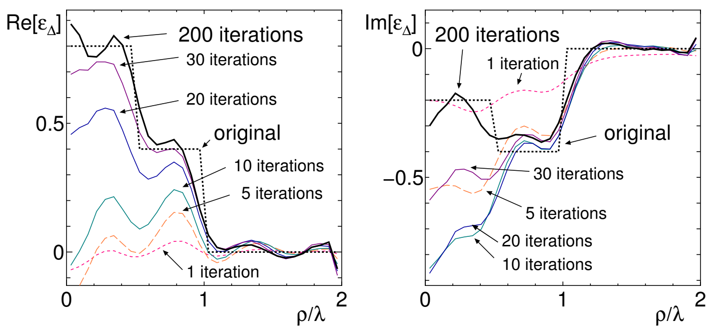

Figure 4: Reconstructed profiles of model B