# ASYMPTOTIC SOLUTION OF DIFFRACTION PROBLEMS IN THE CASES OF WEDGE GEOMETRIES

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Analysis of diffraction problems for wedge-like configuration is of both purely scientific and practical interest. The former is due to the fact that with the exception of the cases of perfect and impedance wedges [1] the closed form solutions are not available. The latter is due to the fact that different wedge-like geometries are often met in many microwave devices and integrated optics devaces. As is well-known the density of electromagnetic energy near edges increases which causes a number of negative effects (overheating, breakdown, etc.) that influence the stability and life-time of devices. The exact static solution for arbitrary wedge geometry involving a perfectly condacting sector [2] allows to find singularity power of the field near the edge (f.n.e.). However it does not allow to estimate the absolute values of f.n.e. for the dynamic case. Meixner's theory [3] developed for the dynamic case does not give new information about the f.n.e. as compared with the static treatment [4].

Thus, to our knowledge theoretical methods that allow to find the analytical relation between the values of f.n.e. and the external fields in the dynamic case are not available at present. Numerical and experimental methods are not convenient for this purpose as a rule. The approach suggested below allows to find this relation and obtain the expressions for the f.n.e. in the analytical form. As an axample, consider the geometry shown in Fig.1. The total field satisfying the wave equation is given by $H_y^t = H_y^s + H_y^i + H_y^r$ - region I; $H_y^t = H_y^s + H_y^T$ - region II, here $H_y^i = H_0 \cdot \exp[ik(z\cdot\sin\theta + x\cdot\cos\theta)]$, $H_y^r = H_0 \cdot \exp[ik\cdot(z\cdot\sin\theta - x\cdot\cos\theta)]$, $H_y^T = TH_0 \cdot \exp[ik\cdot(z\cdot\sin\theta - x\cdot(\varepsilon-\sin^2\theta)^{1/2})]$, $T = 2\cdot\varepsilon\cdot\cos\theta\,[\varepsilon\cdot\cos\theta + (\varepsilon-\sin^2\theta)^{1/2}]^{-1}$, $k = k' - i\,k''$, $0 < k'' \ll k'$, $H_y^s$ is the scattered field. The time dependence $\exp(i\omega t)$ is ommited throughout. By using Fourier transformation of the wave equation and boundary conditions on all the interfaces we can obtain the following singular integral equation [5]:

$$\frac{a(t)}{\lambda(t)} P\int_{-\infty}^{\infty} \frac{\lambda(\tau)\cdot F(\tau)}{\tau - t}\,d\tau + P\int_{-\infty}^{\infty} \frac{F(\tau)}{\tau - t}\,d\tau = \frac{-2\sqrt{2\pi}\,TH_0 k\cdot\sin\theta}{g(t)} \qquad (1)$$

where $a(t)=1+\frac{2\varepsilon\lambda(t)}{\lambda_\varepsilon(t)}$, $g(t)=\lambda_\varepsilon(t)\left[\lambda_\varepsilon(t)+ik\sqrt{\varepsilon-\sin^2\theta}\right]$, $\lambda(t)=\sqrt{t^2-k^2}$, $\lambda_\varepsilon(t)=\sqrt{t^2-k^2\varepsilon}$,

$$F(t) = \Phi(t) - \Phi(-t), \quad \Phi(t) = \left[1/\sqrt{2\pi}\right] \int_{-\infty}^{\infty} H_y^s(z, x=+0) \cdot e^{itz} dz,$$

Re $[\lambda(t), \lambda_\varepsilon(t)] > 0$, Im $[\lambda(t), \lambda_\varepsilon(t)] > 0$, $P$ is the Cauchy's principal value.

Eq (1) is intractable in a closed form. However, the asymptotic solution (1) which is valid for large $t$ ($t \gg k \cdot \varepsilon^{1/2}$) can be obtained. It is $\Phi(t)$ for large $t$ that makes the principal contribution to the f.n.e. formation. By substituting $a(t)$ by its static value $a_o = \lim_{t \to \infty} a(t)$ in (1) the equation obtained can be solved by using an equivalent matrix Hilbert problem [6]:

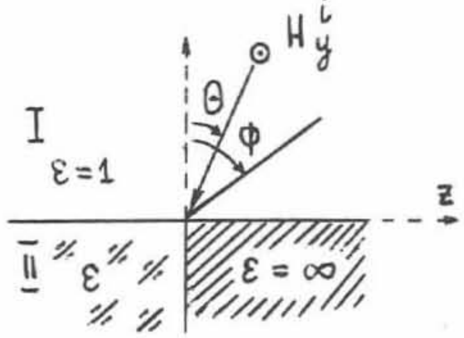


Fig. 1.

$$\vec{\Psi}_+(t) = \hat{G}(t) \cdot \vec{\Psi}_-(t) + f(t), \quad -\infty < t < \infty \tag{2}$$

where $\hat{G}(t) = \dfrac{1}{1+\varepsilon} \begin{bmatrix} \varepsilon & -a_o/\lambda(t) \\ -\lambda(t) & -\varepsilon \end{bmatrix}$, $\vec{\Psi}_\pm(t) = \begin{bmatrix} \Psi_{1\pm}(t) \\ \Psi_{2\pm}(t) \end{bmatrix}$,

$$\Psi_1(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\tau)}{\tau - \zeta} d\tau, \quad \Psi_2(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \lambda(t) \frac{F(\tau)}{\tau - \zeta} d\tau,$$

$$\vec{f}(t) = \sqrt{2/\pi} \; \frac{iTH_0 k \cdot \sin\theta}{(1+\varepsilon) \cdot q(t)} \begin{bmatrix} 1 \\ \lambda(t) \end{bmatrix}, \quad \Psi_{1,2\pm}(t) = \lim_{\mathrm{Im}\zeta \to \pm 0} \Psi_{1,2}(\zeta).$$

$\zeta$ is the complex variable.

By substituting $\vec{\Psi}_+'(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \vec{\Psi}_+(t)$ eq. (2) is reduced to another Hilbert problem with matrix $\hat{G}'(t) = 1/(1+\varepsilon) \begin{bmatrix} \varepsilon & -a_o/\lambda(t) \\ \lambda(t) & \varepsilon \end{bmatrix}$ which allows factorization $\hat{G}'(t) = \hat{X}_+(t) \cdot [\hat{X}_-(t)]^{-1}$ by Khrapkov's method [7]:

$$\hat{X}(\zeta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ch[h(\zeta) \cdot \beta(\zeta)] + \begin{bmatrix} 0 & a_o \\ \lambda^2(t) & 0 \end{bmatrix} \cdot h^{-1}(\zeta) \cdot sh[h(\zeta) \cdot \beta(\zeta)],$$

$$h(\zeta) = i \cdot a_o^{1/2} \lambda(\zeta), \quad \beta(\zeta) = (\tau_o / a_o^{1/2} 2i) \int_{-\infty}^{\infty} \frac{d\tau}{\lambda(\tau) \cdot (\tau - \zeta)},$$

$\tau_0 = (1/\pi)\arccos[\varepsilon/(1+\varepsilon)]$, $\hat{X}_\pm(t) = \lim\limits_{\mathrm{Im}\zeta \to \pm 0} \hat{X}(\zeta)$ . $\hat{X}_\pm(t)$ can be expressed in terms of elementary functions. Further procedure corresponds to the general theory of a matrix Hilbert problem [6]. The only solution corresponding to the edge condition yields

$$\Phi(t) = \sqrt{2/\pi}\,\frac{TH_0 ik\cdot\sin\theta}{1+\varepsilon}\Bigg[-g(t)^{-1} + \frac{1+\varepsilon}{4\pi i\varepsilon}\,P\int_0^\infty \frac{R_1(t)S_1(\tau)\tau\lambda(t) - R_2(t)S_2(\tau)\lambda(\tau)t}{\lambda(t)g(\tau)\cdot(\tau^2-t^2)}\,d\tau \qquad (3)$$

where $R_{1,2}(t) = L(t) \pm L^{-1}(t)$, $S_{1,2}(t) = Q(t) \pm Q^{-1}(t)$,

$L(t) = \left[\dfrac{\lambda(t)-t+k}{\lambda(t)+t-k}\right]^{\tau_0}$, $Q(t) = \left[\dfrac{\lambda(t)-t}{\lambda(t)+t}\right]^{\tau_0/2}$,

$\arg[L(t),Q(t)] \to -\pi$ as $t \to -\infty$

For $\varepsilon = 1$ expression (3) is the exact solution of eq.(1). The proof of its identity with the well-known Sommerfeld solution [1] requires additional investigations and can be made in the nearest future.

For an arbitrary $\varepsilon$ formular (3) can be used for finding the f.n.e.. By using the asymptotic properties of the Cauchy integral and the Fourier integral the values of singular components of the f.n.e. can be estimated:

$$\begin{pmatrix}|E_x|\\|E_z|\end{pmatrix} = \frac{TH_0\sin\theta}{2\pi^2\varepsilon}\left[\frac{2}{k\cdot\rho}\right]^{\tau_0}\Gamma(\tau_0)\,C_1\begin{bmatrix}\cos[\tau_0(\pi/2-\phi)]\\ \sin[\tau_0(\pi/2-\phi)]\end{bmatrix} + O(\rho^{\tau_0}) \qquad (4)$$

where $\rho = (x^2+z^2)^{1/2}$,

$$C_1 = \left|\int_0^\infty [S_1(\tau)\lambda(\tau) + S_2(\tau)\tau]g^{-1}(\tau)\,d\tau\right|,\quad -\pi/2 < \phi < \pi/2,$$

$\Gamma(x)$ is the $\Gamma$-function, the angle $\phi$ is taken as is shown in the Fig.1.

Expression (4) determines the dependence of the singular components of the f.n.e. on the incident field parameters and can be used to eliminate the possibility of a breakdown, overheating, and appearance of other undesired effects. The limitation on the parameters of an incident field is given by the relation $|E(r_0)| < E_{lim}$ , where $E_{lim}$ is the field corresponding to the appearance of the effects mentioned above, $r_0$ is the characteristic radius of the edge curvature under real conditions.

This approach can be reality expended to more complex wedge-like geometries, and the results obtained may be useful in designing different microwave devices and integrated optics devices.

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