# An Asymptotic Series Solution for the Flanged-Waveguide Radiation 

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#### Abstract

The problem of radiation from a flanged parallel plate waveguide is re-examined. The technique of the Fourier transform is used to represent the radiation fields in the spectral domain. The simultaneous equations for the radiation field coefficients are formulated and solved to give an asymptotic series solution. The asymptotic series solution is compared with other results to clarify some ambiguities associated with the numerical inaccuracies among different numerical approaches.


## 1 Introduction

The subject of radiation from a flanged parallel plane waveguide has been long received a great deal of attention due to its important applications to the antenna engineering. Representative theoretical investigations into the flanged waveguide radiation problem are in [1]. Recently, the technique of the moment method [2] has been used to clarify some discrepancies in the numerical results existing among the different techniques as pointed out in [1]. In this paper, we re-examine the problem of the flanged waveguide radiation in the spectral domain which is based on the Fourier transform technique. The objective of this paper is not only to formulate the solution to the flanged-waveguide-radiation problem in an asymptotic series representation but also to compare the series solution to the existing other numerical results. In what follows, we present the expression of the radiation field from the waveguide and compare it with others. A brief summary on the thoeretical development is given in Conclusion.

## 2 Derivation of Radiation Field: $T M^{y}$ Case

Consider a parallel-plate waveguide of width $2 a$ which has an infinite flange as shown in Fig. 1. Assume that a $T M^{y}$ incident field $E_{y}^{i}$ impinges on the aperture from inside the parallel-plate waveguide. Here, $\exp (-j w t)$ time-harmonic variation is assumed throughout.

Then in region (I) of wavenumber $k$, incident and reflected electric fields are respectively written as

$$
\begin{aligned}
& E_{y}^{i}(x, z)=\operatorname{sina} a_{p}(x+a) \exp \left(j \xi_{p} z\right) \\
& E_{y}^{r}(x, z)=\sum_{m=1}^{\infty} c_{m} \sin a_{m}(x+a) \exp \left(-j \xi_{m} z\right)
\end{aligned}
$$

where $\xi_{m}=\sqrt{k^{2}-a_{m}^{2}}, a_{m}=m \pi /(2 a)$
Note that for odd $p, m=1,3,5, \ldots$, and for even $p, m=2,4,6, \ldots$.
In regin (II) of wavenumber $k_{0}$, the transmitted electric field is

$$
\begin{aligned}
E_{y}^{t}(x, z) & =1 /(2 \pi) \int_{-\infty}^{\infty} \tilde{E}_{y}^{t}(\zeta) e^{-j \zeta x+j k_{1} z} d \zeta \\
\tilde{E}_{y}^{t}(\zeta) & =\int_{-\infty}^{\infty} E_{y}^{t}(x, 0) e^{j \zeta x} d x \\
k_{1} & =\sqrt{k_{0}^{2}-\zeta^{2}}
\end{aligned}
$$

Note that $\tilde{E}_{y}^{t}(\zeta)$ and $E_{y}^{t}(x, 0)$ are the Fourier-transformed pair.
Since $H_{x}(x, z)=-1 /(j \omega \mu) \partial E_{y}(x, z) / \partial z$, the corresponding $x$ components of the incident, reflected, and transmitted H-fields are

$$
\begin{aligned}
H_{x}^{i}(x, z) & =-\frac{\xi_{p}}{\omega \mu} \operatorname{sina}_{p}(x+a) \exp \left(j \xi_{p} z\right) \\
H_{x}^{r}(x, z) & =\sum_{m=1}^{\infty} \frac{\xi_{m} c_{m}}{\omega \mu} \sin a_{m}(x+a) \exp \left(-j \xi_{m} z\right) \\
H_{x}^{t}(x, z) & =\frac{-1}{j \omega \mu} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} \frac{1}{2 \pi} \tilde{E}_{y}^{t}(\zeta) e^{-j \zeta x+j k_{1} z} d \zeta
\end{aligned}
$$

To determine unknown coefficient $c_{m}$, it is necessary to match the boundary conditions of tangential E- and H-field continuities. First, the tangential E-field continuity along the irregular boundary ( $-\infty<x<\infty, z=0$ ) yields

$$
\begin{aligned}
E_{y}^{t}(x, 0) & =E_{y}^{i}(x, 0)+E_{y}^{r}(x, 0) & & |x|<a \\
& =0 & & |x|>a
\end{aligned}
$$

Taking the Fourier transform on the both sides of above equation, we get

$$
\begin{align*}
\tilde{E}_{y}^{t}(\zeta) & =\int_{-\infty}^{\infty} E_{y}^{t}(x, 0) e^{j \zeta x} d x  \tag{2.1}\\
& =\int_{-a}^{a}\left[\sin a_{p}(x+a)+\sum_{m=1}^{\infty} c_{m} \sin a_{m}(x+a)\right] e^{j \zeta x} d x \tag{2.2}
\end{align*}
$$

Substituting (2.1) into (2.2), and performing integration with respect to x , we obtain

$$
\begin{align*}
\tilde{E}_{y}^{t}(\zeta) & =K_{p}(\zeta)+\sum_{m=1}^{\infty} c_{m} K_{m}(\zeta)  \tag{2.3}\\
K_{m}(\zeta) & =\frac{a_{m}}{\left(\zeta^{2}-a_{m}^{2}\right)}\left[e^{j \zeta a}(-1)^{m}-e^{-j \zeta a}\right] \tag{2.4}
\end{align*}
$$

Second, the tangential H-field continuity along $-a<x<a, z=0$, gives

$$
\begin{aligned}
H_{x}^{t}(x, 0) & =H_{x}^{i}(x, 0)+H_{x}^{r}(x, 0) \\
\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left[K_{p}(\zeta)+\sum_{m=1}^{\infty} c_{m} K_{m}(\zeta)\right] k_{1} e^{-j \zeta x} d \zeta & =\xi_{p} \sin a_{p}(x+a)-\sum_{m=1}^{\infty} c_{m} \xi_{m} \sin a_{m}(x+a)
\end{aligned}
$$

In order to determine the coefficient $c_{m}$, we multiply the above equation by $\sin a_{n}(x+a)$ and integrate the both sides with respect to $x$ from $-a$ to $a$, then we obtain

$$
\begin{equation*}
\frac{1}{2 \pi}\left[I_{p n}+\sum_{m=1}^{\infty} c_{m} I_{m n}\right]=\xi_{p} a \delta_{n p}-\xi_{n} c_{n} a \tag{2.5}
\end{equation*}
$$

where

$$
I_{m n}=\int_{-\infty}^{\infty} \frac{a_{m} a_{n}\left[(-1)^{m} e^{j \zeta a}-e^{-j \zeta \alpha}\right]\left[(-1)^{n} e^{-j \zeta a}-e^{j \zeta a}\right] k_{1}}{\left(\zeta^{2}-a_{m}^{2}\right)\left(\zeta^{2}-a_{n}^{2}\right)} d \zeta
$$

$I_{m n}$ may be converted into a fast convergent integral using the the technique of the contour integration such as:

$$
I_{m n}=2 \pi a \eta_{m} \delta_{m n}-\left(I_{1}+I_{2}\right)
$$

where $\delta_{m n}$ is the Kronecker delta, and $\eta_{m}=\sqrt{k_{0}^{2}-a_{m}^{2}}$. The detailed evaluation of $I_{m n}$ along with the expressions of $I_{1}$ and $I_{2}$ is given in [3]. The first term containing $\delta_{m n}$ is a residue contribution at $\zeta= \pm a_{m}$ whereas $I_{1}, I_{2}$ arise from the integration along the branch-cut which is associated with the branch-point at $\zeta=k_{0}$. When the operating frequency approaches infinity $\left(k_{0} a \rightarrow \infty\right)$, the branch-cut contribution becomes negligible,thus $I_{m n} \rightarrow 2 \pi a \eta_{m} \delta_{m n}$.

Substituting $I_{m n}$ into (2-5), we obtain the simultaneous equations for $c_{m}$. Solving the simultaneous equations, we may represent the $c_{m}$ in the following matrix form:

$$
C=(U-R)^{-1} Q=Q+R Q+R^{2} Q+\ldots
$$

where $C$ is the column matrix of elements $c_{m}, U$ is the identity matrix, $R$ is the full matrix of elements $r_{n m}$, and $Q$ is the column matrix of elements of $q_{n}$. The expressions of $r_{n m}, q_{n}$ are given as:

$$
\begin{aligned}
r_{n m} & =\frac{\left.\left(I_{1}+I_{2}\right)\right|_{m n}}{2 \pi\left(\xi_{n}+\eta_{n}\right) a} \\
q_{n} & =\frac{\left(\xi_{p}-\eta_{p}\right) \delta_{n p}}{\left(\xi_{n}+\eta_{n}\right)}+\frac{\left.\left(I_{1}+I_{2}\right)\right|_{p n}}{2 \pi a\left(\xi_{n}+\eta_{n}\right)}
\end{aligned}
$$

The examination of $r_{n m}$ reveals that $r_{n m} \sim O\left[1 / \sqrt{k_{0} a}\right]$, for $k_{0} a>1$, and $c_{m}$ may be given as

$$
c_{m}=q_{m}\left(1+O\left[1 / \sqrt{k_{0} a}\right]\right)
$$

In case the flanged waveguide is filled with the air (i.e., $k=k_{0}$ ), then $\xi_{p}=\eta_{p}$, thus

$$
q_{n}=\frac{\left.\left(I_{1}+I_{2}\right)\right|_{p n}}{2 \pi a\left(\xi_{n}+\eta_{n}\right)} \sim O\left(1 / \sqrt{k_{0} a}\right)
$$

This means that the reflection coefficient, $\left|c_{m}\right|^{2}$ is of the order $\left(1 /\left(k_{0} a\right)\right)$. Fig. 2 shows the input impedance $Z$ versus $2 a / \lambda$ when $p=1$, where $Z=\left[\left(1+c_{1}\right) /\left(1-c_{1}\right)\right]^{*}$. Note that our computation well agrees with the results in [2], thus reconfirming the numerical accuracy of [1].

## 3 Concluding Remarks

Using the Fourier transform approach, we obtain the asymptotic series solution for the radiation problem of the flanged waveguide. The series solution is compared with several existing approximate results in order to clarify some ambiguities associated with numerical inaccuracies in the computation of the reflection coefficients.

## References

[1] S.W. Lee and L. Grun, "Radiation from flanged waveguide: Comparison of solutions," IEEE Trans. on Antennas and Propagt. vol. 30, no. 1, Jan. 1982
[2] M.S. Leong, P.S. Kooi, and Chandra, "Radiation from a flanged parallel-plate waveguide: Solution by moment method with inclusion of edge condition", IEE Proceedings, vol. 135, Pt. H, no. 4, Aug. 1988, pp. 249-255
[3] H.J. Eom and T.J. Park, and K. Yoshitomi,"An analysis of TM-scattering from a rectangular channel in a conducting plane," submitted to the IEEE Trans. on Antennas and Propagt. Oct. 1991


Fig. 1: Geometry of Radiation problem Fig. 2: Behavior of Input Impedance versus $2 a / \lambda$.

