MOMENT METHOD SOLUTION OF THE TIME-DEPENDENT TRANSPORT EQUATION

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ABSTRACT

The time dependent transport equation is used to study the transmission of periodic pulses through a scattering environment. The transport equation is solved by the moment method using triangular expansion functions. Numerical results indicate the dependence of the incoherent intensity on penetration depth, observation angle, albedo, pulse penetration depth and pulse repetition rate.

1. INTRODUCTION

A gaussian plane wave pulse train of time period T seconds and propagating through free space is assumed to be normally incident upon a statistically homogeneous, isotropically scattering half-space region characterized by absorption (σ_a) and scatter (σ_s) cross-sections per unit volume. Transport theory is used to track the flow of energy through this environment. Below, the time dependent equation for plane-parallel regions is introduced. The moment method is then used to reduce this integro-differential equation to a system of linear equations. Finally, numerical results are presented.

2. FORMULATION

The intensity I_o of the incident gaussian pulse train is chosen to be an even function of time and hence is expressed as a cosine Fourier series:

$$I_o(z',t';\mu,\phi) = \bar{S}_o f(z',t')\delta(\mu-\mu_o)\delta(\phi-\phi_o), \mu = \cos\theta, \mu_o = \cos\theta_o, \quad (1a)$$

$$f(z',t') = \mathrm{Re}\sum_{\nu=0}^{\infty} f_\nu e^{j\nu\omega'(t'-z')}, \omega' = 2\pi/T', T' = \sigma_t cT, \quad (1b)$$

where normalized coordinates $z'=\sigma_t z, t'=\sigma_t ct$ are used, $\sigma_t=\sigma_s+\sigma_a$ and c is the speed of light in vacuum; $\bar{S}_o f(z',t')$ is the incident instantaneous power density, $\bar{S}_o$ is the average incident power per unit area in time period T', angles $\theta_o=\phi_o=0$ for normal incidence, f_ν is real and $\delta(x)$ is the Dirac delta function. For $-T'/2 \le t' \le T'/2$ and in the $z'=0$ plane, the incident gaussian pulse shape is expressed as

$$f(o,t') = \frac{\alpha}{\sqrt{\pi}} e^{-(\alpha t'/T')^2}, \quad \frac{1}{T}\int_{-T/2}^{T/2} f(o,t')dt' = 1, \quad \alpha = \text{constant}. \quad (2)$$

Assuming $f(0,t') \to 0$ as $t' \to \pm T'/2$, Fourier coefficients in (1b) become

$$f_\nu = \epsilon_\nu e^{-(\pi\nu/\alpha)^2}, \quad \nu = 0,1,2,\dots \; ; \; \epsilon_\nu = 1, \nu=0; \; \epsilon_\nu = 2, \nu \neq 0 . \tag{3}$$

Allowing intensity to be the sum of two components - the coherent intensity I_c and the incoherent or diffuse intensity I_d- and considering only the latter, the time-dependent transport equation for normal incident and in plane - parallel geometry takes the form [1]

$$\frac{\partial I_d}{\partial t'} + \mu\frac{\partial I_d}{\partial z'} + I_d(z',t';\mu) = \frac{W}{2}\int_{-1}^{1} I_d \, d\mu' + \frac{W\bar{S}_o}{4\pi} e^{-z'} f(z',t'), \quad z' \geqslant 0 . \tag{4}$$

Note that symmetry requires that I_d be independent of ϕ and albedo $W = \sigma_s/\sigma_t$. Boundary conditions which must be satisfied by the solution to (4) are

$$\text{(a)} \; I_d(0,t';\mu) = 0, \quad 0 \leq \mu \leq 1 \qquad \text{(b)} \; I_d \to 0 \quad \text{as } z' \to \infty . \tag{5}$$

The first condition in (5) results because I_d can only leave the scatter domain; the second condition appears because of absorption losses.

3. MOMENT METHOD SOLUTION

Let I_d be represented by the truncated series expansion

$$I_d(z',t';\mu) = \sum_{n=0}^{N} I_n(z',t')\, q_n(\mu) , \tag{6a}$$

where $I_n(z',t')$ are the unknown expansion coefficients and $q_n(\mu)$ are triangular expansion functions

$$q_n(\mu) = \begin{cases} \pm \frac{1}{\Delta\mu}(\mu - \mu_{n\mp1}), & \begin{matrix}\mu_{n-1} \\ \mu_n\end{matrix} \leq \mu \leq \begin{matrix}\mu_n \\ \mu_{n+1}\end{matrix}, \quad n \neq 0, N \\ 0 & , \text{elsewhere} \end{cases} \tag{6b}$$

$$q_o(\mu) = \begin{cases} -\frac{1}{\Delta\mu}(\mu - \mu_1), & \mu_o = -1 \leq \mu \leq \mu_1, \; n=0 \\ 0 & , \text{elsewhere} \end{cases} ; \; q_N(\mu) = \begin{cases} \frac{1}{\Delta\mu}(\mu - \mu_{N-1}) & \mu_{N-1} \leq \mu \leq \mu_N = 1, n=N \\ 0 & , \text{elsewhere} \end{cases} \tag{6c}$$

where $\Delta\mu = \mu_n - \mu_{n-1} = 2/N$ and $\mu_n = (2n-N)/N$. Restricting scatter angles to those specified by $\mu \doteq \mu_m$, $m=0,\dots,N$ and letting

$$I_m(z',t') = \mathrm{Re} \sum_{\nu=0}^{N} I_{m\nu}(z') e^{j\nu\omega'(t'-z')} \tag{7}$$

yields from (4) the following system of (N+1) equations for (N+1) unknowns $I_{m\nu}(z)$, $\nu=0,1,\dots N$,:

$$[1 + j\nu\omega'(1-\mu_m)] I_{m\nu}(z') + \mu_m \frac{dI_{m\nu}}{dz'} = \frac{W}{4}\Delta\mu \sum_{n=0}^{N} \epsilon_n I_{n\nu}(z') + \frac{W\bar{S}_o}{4\pi} e^{-z'} f_\nu \tag{8}$$

subject to boundary conditions of all ν

$$\text{(a)}\ I_{m\nu}(0)=0,\ \frac{N+1}{2}\leq m\leq N \qquad \text{(b)}\ I_{m\nu}(z')\to 0 \ \text{ as } z'\to\infty \ \text{for all } m \quad (9)$$

The homogeneous and particular solutions to (8) and (9) are deduced to be

$$I_{m\nu}(z')=I^{h}_{m\nu}(z')+I^{p}_{m\nu}(z')=\sum_{k=\frac{N+1}{2}}^{N}\frac{Q_{\nu k}\,e^{-z'/s_{\nu k}}}{\left[1+j\nu\omega'(1-\mu_m)-\frac{\mu_m}{s_{\nu k}}\right]}-\frac{\bar{S}_0 N}{2\pi}f_\nu e^{-z'}\delta_{mN}, \quad m=0,1,\ldots,N \qquad (10a)$$

where $s_{\nu k}$ are eigenvalues determined from the characteristic equation

$$1=\frac{W}{N}\sum_{n=2}^{N}\frac{\epsilon_n}{2}\,\frac{1}{1+j\nu\omega'(1-\mu_n)-\frac{\mu_n}{s}}, \quad \epsilon_n=\begin{cases}1, & n=0,N\\ 2, & 0<n<N\end{cases} \qquad (10b)$$

with $R_e\ s_{\nu k}>0$, $k=\frac{N+1}{2},\ldots N$ so that (9b) is satisfied and where (9a) yields the system of equations for the unknown constants $Q_{\nu k}$; δ_{mN} is the Kronecker delta function.

4. NUMERICAL RESULTS

In Fig. 1, diffuse intensity is plotted versus penetration depth z'. Initially, the diffuse intensity increases. After depth Z'=1.0, I_d decreases (not shown). A comparison of Figs. 1 and 2 indicates that as the period T' increases the pulse shape becomes more distorted. This indicates that adjacent pulses of the signal with the larger period tend to overlap more strongly than when T' is smaller. In Figs. 3 and 4, I_d is shown at z'=0.1 in eight directions. Observe that at small depth intensity is strongest near the vertical direction ($\theta=87.3^{o}$). Observe also that for $\theta>90^{o}$ I_d remains approximately uniform in time.

5. REFERENCES

1. Weinberg, A. and Wigner, E., The Physical Theory of Newton Chain Reactors, Univ. of Chicago Press, (1958).

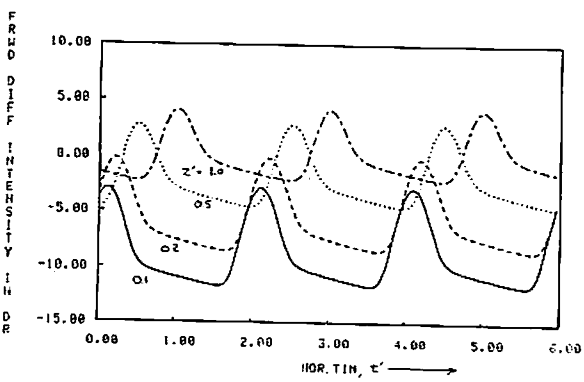

Fig.1. FORWARD DIFFUSE INTENSITY FOR 1/2 SPACE, ξ = 4π, W=0.90, T'=2.0, θ=0°

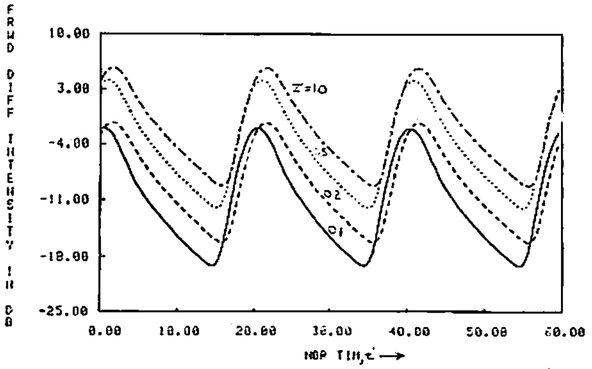

Fig. 2. FORWARD DIFFUSE INTENSITY FOR 1/2 SPACE, $\bar{S}_0 = 4\pi$, W=0.90, T'=20, θ=0°

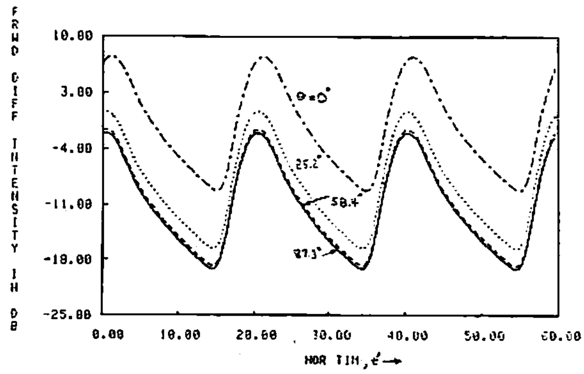

Fig. 3. FORWARD DIFF INTENSITY FOR 1/2 SPACE, $\bar{S}_0$ =4*PI, W=0.90, T'=20, Z'=0.1

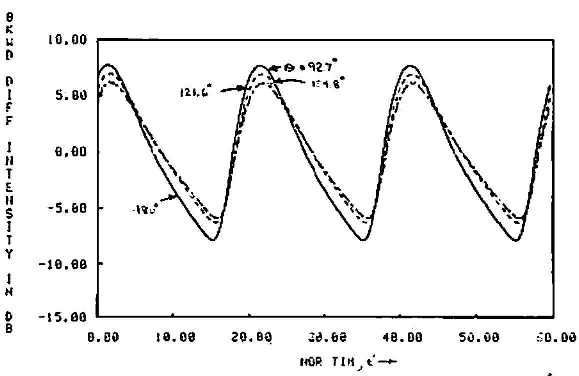

Fig. 4. BACKWARD DIFFUSED INTENSITY FOR 1/2 SPACE, $\bar{S}_0$ =4*PI, W=0.90, T'=20, Z'=0.1