Z-Transform PML Algorithm for Truncating Metamaterial FDTD Domains

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***Abstract*— Efficient and simple formulations of the Perfectly Matched Layer (PML) are presented for truncating metamaterial Finite Difference Time Domain (FDTD) grids. The formulations are based on incorporating the Z-transform theory into the FDTD algorithm to model the frequency dependence property of the metamaterials. Numerical example carried out in one dimensional metamaterial domain is included to validate the proposed formulations.**

***Index Terms*— Metamaterial, finite difference time domain, perfectly matched layer, Z-transform.**

I. Introduction

In recent years, the electromagnetic metamaterials with simultaneously negative electric permittivity and magnetic permeability [1] have been received much attention due to their unusual electromagnetic properties. These materials, referred as Negative Index Materials (NIM), have been realized by using different models such as the cold plasma, Drude medium and Lorentzian medium models. Recently, the Finite Difference Time Domain (FDTD) method [2] has been used successfully in simulating electromagnetic wave propagation in domains that have metamaterial properties [3]-[8].

When the FDTD method is used for modelling open region metamaterial domains, efficient Absorbing Boundary Conditions (ABCs) are needed to truncate the computational domain. Berenger's Perfectly Matched Layer (PML) [9] has been shown to be one of the most effective FDTD ABCs. Very recently, different PML formulations have been successfully introduced for truncating metamaterial FDTD computational domains [10]-[12]. In these formulations, the auxiliary differential equation (ADE) method and the bilinear frequency approximation technique have been incorporated into the FDTD implementations of the metamaterial.

In this paper, alternative PML formulations are presented for truncating metamaterial domains. The formulations incorporate the Z-transform theory [13] into the FDTD implementation of the frequency dependence of the metamaterials. The method has the advantage of simplicity as it allows direct FDTD implementations Maxwell's equations in the metamaterial domains. Numerical example carried out in one-dimensional (1-D) domain composed entirely of Lorentzian type metamaterial is included to validate the proposed formulations.

The paper is organized as follows. In section 2, the formulations of the proposed method are presented. In section 3, numerical results are included to validate the proposed formulations. Finally, conclusions are included in section 4.

II. Formulation

Using the PML formulations of [12], the normalized field equations for a $z-$polarized transverse electromagnetic (TEM) wave propagating through metamaterial domain along the $x-$direction can be written in the frequency domain as

$$j\omega\varepsilon_r(\omega)S_x\widetilde{E}_z = c\frac{\partial}{\partial x}\widetilde{H}_y \quad (1)$$

$$j\omega\mu_r(\omega)S_x\widetilde{H}_y = c\frac{\partial}{\partial x}\widetilde{E}_z \quad (2)$$

where $\widetilde{E}_z$, and $\widetilde{H}_y$ are the Fourier transform of E_z and H_y fields, respectively, $\varepsilon_r(\omega)$ and $\mu_r(\omega)$ are, respectively, the permittivity and permeability of the metamaterial, and S_x is the PML parameter defined as [12]

$$S_x = 1 + \frac{\sigma_x}{j\omega\varepsilon_o\varepsilon_r(\omega)} \quad (3)$$

where σ_x is the PML conductivity profile along the $x-$direction. In this paper, $\varepsilon_r(\omega)$ and $\mu_r(\omega)$ are assumed to be identical and realized with a Lorentz medium model [4] given by the following expression:

$$\varepsilon_r(\omega) = \mu_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\Gamma} \quad (4)$$

where ω_p is the plasma frequency, ω_0 is the resonance frequency, and Γ is the absorbtion parameter of the medium. Substituting (3) and (4) into (1) and (2), the following can be obtained

$$j\omega\widetilde{E}_z + j\omega\widetilde{D}_z + \frac{\sigma_x}{\varepsilon_o}\widetilde{E}_z = c\frac{\partial}{\partial x}\widetilde{H}_y \quad (5)$$

$$j\omega\widetilde{H}_y + j\omega\widetilde{B}_y + \frac{\sigma_x}{\varepsilon_o}\widetilde{H}_y = c\frac{\partial}{\partial x}\widetilde{E}_z \quad (6)$$

where $\widetilde{D}_z$ and $\widetilde{B}_z$ are given by

$$\widetilde{D}_z = \kappa(\omega)\widetilde{E}_z \quad (7)$$

$$\widetilde{B}_y = \kappa(\omega)\widetilde{H}_y \quad (8)$$

where $\kappa(\omega)$ is given by

$$\kappa(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\Gamma} \tag{9}$$

By using the Fourier transform relation $j\omega \rightarrow \partial/\partial t$, (5) and (6) can be written in the time domain as

$$\frac{\partial}{\partial t}E_z + \frac{\partial}{\partial t}D_z + \frac{\sigma_x}{\varepsilon_o}E_z = c\frac{\partial}{\partial x}H_y \tag{10}$$

$$\frac{\partial}{\partial t}H_y + \frac{\partial}{\partial t}B_y + \frac{\sigma_x}{\varepsilon_o}H_y = c\frac{\partial}{\partial x}E_z \tag{11}$$

Equations (10) and (11) can be discretized by using the FDTD algorithm [2] as

$$\begin{aligned} E_{z_i}^{n+1} &= r_{0_i}E_{z_i}^n - r_{1_i}\left(D_{z_i}^{n+1} - D_{z_i}^n\right) \\ &+ r_{1_i}\chi\left(H_{y_{i+1/2}}^{n+1} - H_{y_{i-1/2}}^{n+1}\right) \end{aligned} \tag{12}$$

$$\begin{aligned} H_{y_{i+1/2}}^{n+1} &= r_{0_{i+1/2}}H_{y_{i+1/2}}^n - r_{1_{i+1/2}}\left(B_{y_{i+1/2}}^{n+1} - B_{y_{i+1/2}}^n\right) \\ &+ r_{1_{i+1/2}}\chi\left(E_{z_{i+1}}^{n+1} - E_{z_i}^{n+1}\right) \end{aligned} \tag{13}$$

where Δ_t is the time step, Δ is the space cell size, $\chi = c\Delta_t/\Delta$ and

$$r_{0_i} = \frac{1-p_i}{1+p_i}, \text{ and } r_{1_i} = \frac{1}{1+p_i} \tag{14}$$

with

$$p_i = \frac{\sigma_{x_i}\Delta_t}{2\varepsilon_0} \tag{15}$$

In (12) and (13), D_z^{n+1} and B_y^{n+1} can be computed easily from (7) and (8), respectively, by using the Z-transform theory [13]. Hence, (9) can be written in the Z-domain [13] as

$$\kappa(\mathcal{Z}) = \frac{b_0\mathcal{Z}^{-1}}{1 + a_0\mathcal{Z}^{-1} + a_1\mathcal{Z}^{-2}} \tag{16}$$

where $\mathcal{Z}$ is the $\mathcal{Z}-$transform variable and a_0, a_1 and b_0 are given by

$$\begin{aligned} a_0 &= -2e^{-\alpha\Delta_t}\cos(\beta\Delta_t),\ a_1 = e^{-2\alpha\Delta_t}, \\ b_0 &= \gamma e^{-\alpha\Delta_t}\sin(\beta\Delta_t) \end{aligned} \tag{17}$$

with

$$\alpha = \frac{\Gamma}{2},\ \beta = \sqrt{\omega_0^2 - \alpha^2}, \text{ and } \gamma = \frac{\Delta_t\omega_p^2}{\beta} \tag{18}$$

Using (16) and taking the Z-transform of (7) and (8), the discretization of D_z and B_y can be obtained easily by using the Z-transform relation $\mathcal{Z}^{-m}W(\mathcal{Z}) \rightarrow W^{n-m}$ as

$$D_{z_i}^{n+1} = -a_0D_{z_i}^n - a_1D_{z_i}^{n-1} + b_0E_{z_i}^n \tag{19}$$

$$B_{y_{i+1/2}}^{n+1} = -a_0B_{y_{i+1/2}}^n - a_1B_{y_{i+1/2}}^{n-1} + b_0H_{y_{i+1/2}}^n \tag{20}$$

It should be pointed out that the proposed formulations are applied in the PML regions at the domain boundaries. In non-PML regions, i.e., in the inner FDTD domain, it is only required to set the coefficients r_0 and r_1, defined in (14), to unity.

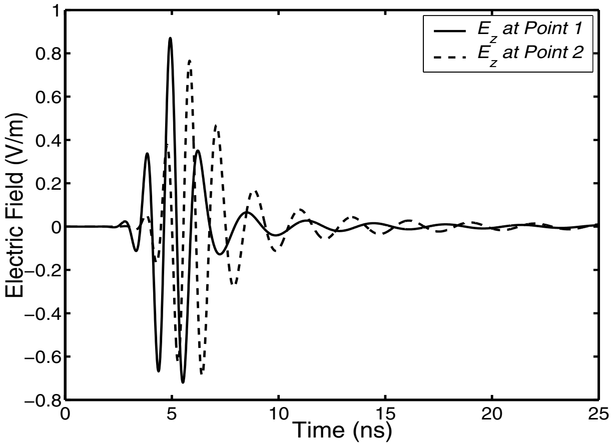

Fig. 1. Early time domain response of E_z as observed 20 cells (Point 1) and 40 cells (Point 2) away from the excitation point.

III. Simulation Study

To show the validity of the proposed formulations, a $z-$polarized Gaussian pulse, with a central radial frequency equals to $\omega_c = 5.0 \times 10^9 rad/s$, was excited at the center of 1-D computational domain which extends in the x direction and entirely composed of metamaterial realized with a Lorentz model [4] with the parameters of $\omega_0 = 1.0 \times 10^9 rad/s$, $\omega_p = 6.9282 \times 10^9 rad/s$ and $\Gamma = 0$. With these choices, $\varepsilon_r(\omega) = \mu_r(\omega) = -1$ at the peak of the incident spectrum ω_c. The size of the computational domain was $100\Delta x$ where $\Delta x = 5mm$. In this test, the time step was chosen as $\Delta_t = 8.33ps$ and the simulation was carried out for the first 32768 time steps.

Figure 1 show the time domain response of the E_z field recorded at two points: point 1 is located 20 cells and point 2 is located 40 cells away from the excitation point. Both ends of the computational domain were terminated by eight additional PML layers with a quadratic conductivity profile and with 0.001% theoretical reflection coefficient, i.e., PML[8, 2, 0.001%], as defined in [9]. As can be seen from Fig. 1, the early time response of the E_z field at point 1 occurs before the response at point 2. This confirms that the metamaterial satisfies the causality principle [3]. On the other hand, it is interesting to note that the response at point 2 starts to lead the response at point 1 in the late time as can be seen in Fig. 2 (an expanded view of Fig. 1 in the late time). This is due to the negative index properties of the metamaterial [3].

To measure the reflection performance of the proposed formulations, a reference field is needed. In this paper, the reference field was computed by using a larger domain with the size of $1000\Delta x$ and truncated by 32 PML layers with the parameters of PML[32, 4, 0.001%]. Figure 3 shows the relative reflection error of the proposed PML formulations as observed one cell away from the PML/computational domain interface for four and eight PML layers with the parameters of PML[4, 2, 0.001%] and PML[8, 2, 0.001%]. The reflection

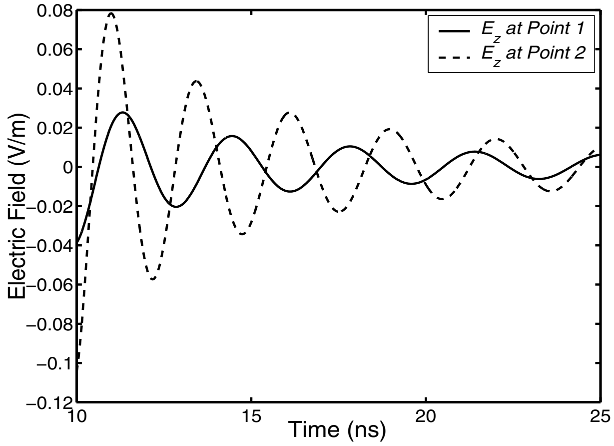

Fig. 2. Late time domain response of E_z as observed 20 cells (Point 1) and 40 cells (Point 2) away from the excitation point.

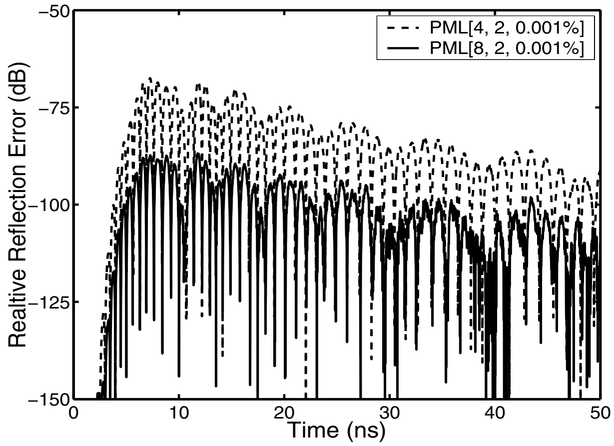

Fig. 3. Relative reflection error of the proposed PML as a function of time.

error was computed as

$$\mathcal{R}_{dB} = 20 \log_{10} \left(\frac{\left| E_z^R(t) - E_z^T(t) \right|}{\max \left[\left| E_z^R(t) \right| \right]} \right) \tag{21}$$

where $E_z^T(t)$ is the field computed using the test domain, and $E_z^R(t)$ is the reference field computed using the larger domain. It is apparent from Fig. 3 that good absorbing performance of the proposed formulations has been achieved. Finally, the reflection performance of the proposed formulations as function of frequency was studied. Figure 4 shows the frequency spectrum of the PML reflection coefficient as obtained by using the proposed formulations and by the PML formulations of [12]. The reflections coefficient was observed one space cell away from the PML/computational domain interface and computed as

$$\mathcal{R}_{dB}(f) = 20 \log_{10} \left| \frac{\mathcal{F}\left\{ E_z^R(t) - E_z^T(t) \right\}}{\mathcal{F}\left\{ E_z^R(t) \right\}} \right| \tag{22}$$

where $\mathcal{F}\{.\}$ is the Fourier transform operation. Similar to results reported a single above, the proposed formulations

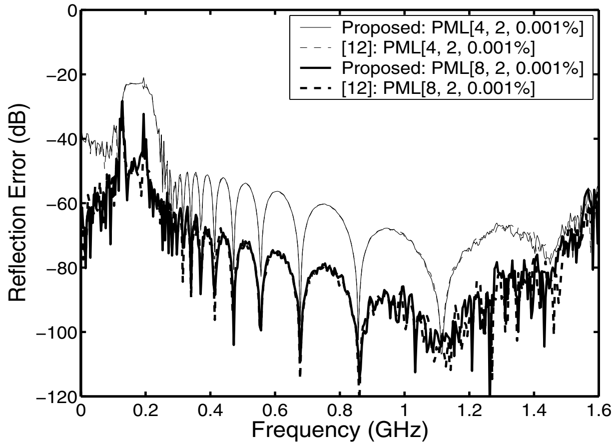

Fig. 4. Reflection coefficient of the proposed PML formulations as a function of frequency.

give good absorbing performance over the frequency range of interest. Furthermore, it should be mentioned that the results of the proposed formulations are very similar to the results obtained by using the PML formulations introduced in [12]. Also, it must be noted that there a sharp increase in the reflection coefficient in the frequency range of 0.1-0.3 GHz, but this is mainly due to the resonance frequency of the metamaterial, i.e., $f_0 = \omega_0/2\pi = 0.15915GHz$. Finally, it should be mentioned that the reflections start to increase for frequencies beyond 1.60 GHz. This is because the pulse used in this paper does not contain significant frequency components at those frequencies.

IV. Conclusion

In this paper, efficient and simple PML formulations are presented for truncating metamaterial computational domains. In the proposed formulations, the frequency dependence of the metamaterials has been implemented in the FDTD algorithm by using the Z-transform theory. Numerical example carried out in one dimensional domain composed entirely of Lorentzian type metamaterial show that good absorbing performance has been obtained with the proposed PML formulations. The formulations can be extended to 2-D and 3-D cases in a similar manner.

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