2E1-4

On Accuracy of Extrapolated Absorbing Boundary Condition

Kazunori Uchida [1], Xue-Feng Li [1], Hiroshi Maeda [1], Kwang-Yeol Yoon [2]
[1] *Department of Communication and Information Engineering, Fukuoka Institute of Technology*
Fukuoka-shi, 811-0295, Japan, k-uchida@fit.ac.jp
[2] Keimyung University, Korea

1. Introduction

With the rapid development of high speed and large memory computers, the finite difference time domain (FDTD) method have been widely used, since Yee first proposed the algorithm [1]. Because of the finite memory size of computer, we have to realize a virtual computational space by introducing absorbing boundary conditions (ABCs) [2]. In earlier times, Mur's ABCs [3] were used by many researchers, but it is inevitable for this type of ABCs that some amount of fictitious reflections are always observed. As is well known, PML was successfully introduced by Berenger [4] to overcome this difficult situation.

ABCs based on PML are excellent, but a lot of computer memories are needed to implement them on computers. To save computer memories, we have proposed EABCs in a compact form [5], where the boundary fields are extrapolated in terms of the linear combination of the fields at two inner points adjacent to the boundary. We have also discussed the EABCs [6] from a view point of the CIP method [7], and it has been demonstrated that the accuracy of the 1D EABCs is excellent and that 2D and 3D EABCs require much smaller memories than PML. In those EABCs, however, we have used the same decomposed field components as PML in order to extrapolate the boundary fields from those in two cell layers adjacent to the boundaries, resulting in computational complexity and memory consuming.

In this paper, we discuss the basic theory of the EABCs in relation to the CIP method, and we propose a new version of EABCs associated with the FVTD method [8]. In the present EABCs, we use only the ordinary electromagnetic field components in the computational region. We also show some numerical examples to check the accuracy of the present EABC method.

2. Theory

2.1 Maxwell's Equations

The Maxwell's equations are written as follows:

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{c\partial t}\tilde{\boldsymbol{H}} \;, \quad \nabla \times \tilde{\boldsymbol{H}} = +\frac{\partial}{c\partial t}\boldsymbol{E} \tag{1}$$

where $c = 1/\sqrt{\epsilon\mu}$ is the light velocity in a homogenious dielectric and magnetic medium which is assumed to be lossless. Moreover, for a computational reason, the magnetic field is normalized by the intrinsic impedance of the medium as indicated by $\tilde{\boldsymbol{H}} = \sqrt{\mu/\epsilon}\boldsymbol{H}$. Adding and subtracting each field component in the Maxwell's equations, we obtain the advection type of equations with non-advection terms in the right hand sides as follows:

$$\begin{aligned}
&\frac{\partial}{c\partial t}(E_y \pm \tilde{H}_z) \pm \frac{\partial}{\partial x}(E_y \pm \tilde{H}_z) = \pm\frac{\partial}{\partial y}E_x + \frac{\partial}{\partial z}\tilde{H}_x \;, \quad \frac{\partial}{c\partial t}(E_z \mp \tilde{H}_y) \pm \frac{\partial}{\partial x}(E_z \mp \tilde{H}_y) = \pm\frac{\partial}{\partial z}E_x - \frac{\partial}{\partial y}\tilde{H}_x \\
&\frac{\partial}{c\partial t}(E_z \pm \tilde{H}_x) \pm \frac{\partial}{\partial y}(E_z \pm \tilde{H}_x) = \pm\frac{\partial}{\partial z}E_y + \frac{\partial}{\partial x}\tilde{H}_y \;, \quad \frac{\partial}{c\partial t}(E_x \mp \tilde{H}_z) \pm \frac{\partial}{\partial y}(E_x \mp \tilde{H}_z) = \pm\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial z}\tilde{H}_y \\
&\frac{\partial}{c\partial t}(E_x \pm \tilde{H}_y) \pm \frac{\partial}{\partial z}(E_x \pm \tilde{H}_y) = \pm\frac{\partial}{\partial x}E_z + \frac{\partial}{\partial y}\tilde{H}_z \;, \quad \frac{\partial}{c\partial t}(E_y \mp \tilde{H}_x) \pm \frac{\partial}{\partial z}(E_y \mp \tilde{H}_x) = \pm\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial x}\tilde{H}_z
\end{aligned} \tag{2}$$

It should be noted that the above advection type of equations are expressed in terms of ordinary field components contrary to the former version where the same decomposed field components as PML were used [6].

2.2 CIP Method

All equations in Eq.(2) are reduced to one of the two advection type of equations with non-advection terms in the right hand sides as follows:

$$\frac{\partial}{c\partial t}\Phi_p + \frac{\partial}{\partial \xi}\Phi_p = \frac{\partial}{\partial \eta}\Psi_p \;, \quad \frac{\partial}{c\partial t}\Phi_m - \frac{\partial}{\partial \xi}\Phi_m = \frac{\partial}{\partial \eta}\Psi_m \tag{3}$$

ISBN: 978-4-88552-223-9 C3055©IEICE

where ξ and η correspond to x, y or z and $\xi \neq \eta$. The first equation shows advection in the plus ξ-direction (p), and the second shows advection in the minus ξ-direction (m). According to the CIP method [7], we can solve these two advection type of equations approximately by the method of time-splitting described in the subsequent discussion.

First, neglecting the non-advection terms in Eq.(3), we obtain the following advection equations:

$$\frac{\partial}{c\partial t}\Phi_p + \frac{\partial}{\partial \xi}\Phi_p = 0 \ , \quad \frac{\partial}{c\partial t}\Phi_m - \frac{\partial}{\partial \xi}\Phi_m = 0 \tag{4}$$

As is well-known, the solutions of the above advection equations can be expressed as $\Phi_p = f(ct - \xi)$ and $\Phi_m = g(ct + \xi)$ where $f(\xi)$ and $g(\xi)$ are arbitrary continuous differentiable functions with respect to ξ. As a result, the value of Φ_p at $t = \tau + \Delta t$ and $\xi = a$ agrees with that at $t = \tau$ and $\xi = a - c\Delta t$. Similarly, the value of Φ_m at $t = \tau + \Delta t$ and $\xi = a$ agrees with that at $t = \tau$ and $\xi = a + c\Delta t$. If $\Phi_{p,m}$ are given in a discrete form, we can estimate their values in terms of interpolation. It should be noted that two values at one point, that is, field intensity and its derivative, are used for interpolation in the CIP formulation. In case of FDTD or FVTD, however, it is not easy to use the derivatives of fields, since it causes memory consuming or poor accuracy. In this paper, we use the Lagrange's interpolation for CIP computations to extrapolate the boundary fields. Now we discretize the functions in Eq.(3) both in time and space as follows:

$$\Phi^n_{p,m}(i,j) = \Phi_{p,m}(n\Delta t, i\Delta\xi, j\Delta\eta) \ , \quad n = 0, 1, 2, \cdots \ , \quad i, j = 0, \pm 1, \pm 2, \cdots \tag{5}$$

where the relation $c\Delta t = \Delta\xi = \Delta\eta$ is assumed. Then, as stated before, CIP method for the advections in Eqs.(4) leads to the following solutions:

$$\Phi^{n*}_p(i,j) = \Phi^n_p(i,j) + \Delta^{(+\xi)}_{adv}(c\Delta t; 0,1,2)\Phi^n_p(i,j) \ , \ \Phi^{n*}_m(i,j) = \Phi^n_m(i,j) + \Delta^{(-\xi)}_{adv}(c\Delta t; 0,1,2)\Phi^n_m(i,j) \tag{6}$$

where $n*$ is an appropriate time-step parameter within $n+1 > n* > n$ [7]. Moreover, the increments of advections in positive and negative ξ-directions are evaluated in terms of the Lagrange's interpolation as follows:

$$\begin{aligned} \Delta^{(+\xi)}_{adv}(\alpha; k_1, k_2, k_3)\Phi^n_p(i,j) &= \sum_{k=k_1,k_2,k_3} w_k(\alpha)[\Phi^n_p(i-k,j) - \Phi^{n-1}_p(i-k,j)] \\ \Delta^{(-\xi)}_{adv}(\alpha; k_1, k_2, k_3)\Phi^n_m(i,j) &= \sum_{k=k_1,k_2,k_3} w_k(\alpha)[\Phi^n_m(i+k,j) - \Phi^{n-1}_m(i+k,j)] \end{aligned} \tag{7}$$

where $k_1 = 0$, $k_2 = 1$ and $k_3 = 2$ are assumed, and the Lagrange's weights at the three points are given by

$$w_k(\alpha) = {\prod}'_{\ell=k_1,k_2,k_3} \frac{(\alpha - \ell\Delta\xi)}{(k\Delta\xi - \ell\Delta\xi)} \ , \quad (k = k_1, k_2, k_3 \ \text{ and } \ \text{k} \neq \ell) \tag{8}$$

where $\prod'$ indicates that the product at $\ell = k$ should be excluded.

Second, based on the time-splitting approximation [7], we include the non-advection terms in Eq.(3) by solving the following equations:

$$\frac{\partial}{c\partial t}\Phi_p = \frac{\partial}{\partial \eta}\Psi_p \ , \quad \frac{\partial}{c\partial t}\Phi_m = \frac{\partial}{\partial \eta}\Psi_m \tag{9}$$

Taking differences both in time and space domains leads to the following approximate solutions:

$$\Phi^{n+1}_p(i,j) = \Phi^{n*}_p(i,j) + \Delta^{(\eta)}_{non}(c\Delta t)\Psi^n_p(i,j) \ , \ \ \Phi^{n+1}_m(i,j) = \Phi^{n*}_m(i,j) + \Delta^{(\eta)}_{non}(c\Delta t)\Psi^n_m(i,j) \tag{10}$$

where

$$\begin{aligned} \Delta^{(\eta)}_{non}(c\Delta t)\Psi^n_p(i,j) &= \frac{c\Delta t}{2\Delta\eta}[\Psi^n_p(i,j+1) - \Psi^n_p(i,j-1)] \\ \Delta^{(\eta)}_{non}(c\Delta t)\Psi^n_m(i,j) &= \frac{c\Delta t}{2\Delta\eta}[\Psi^n_m(i,j+1) - \Psi^n_m(i,j-1) \end{aligned} \tag{11}$$

Third, combining the above two results, we obtain the approximate CIP solutions to the advection type of equations in Eq.(3) as follows:

$$\begin{aligned} \Phi^{n+1}_p(i,j) &= \Phi^n_p(i,j) + \Delta^{(+\xi)}_{adv}(c\Delta t; 0,1,2)\Phi^n_p(i,j) + \Delta^{(\eta)}_{non}(c\Delta t)\Psi^n_p(i,j) \\ \Phi^{n+1}_m(i,j) &= \Phi^n_m(i,j) + \Delta^{(-\xi)}_{adv}(c\Delta t; 0,1,2)\Phi^n_m(i,j) + \Delta^{(\eta)}_{non}(c\Delta t)\Psi^n_m(i,j) \end{aligned} \tag{12}$$

2.3 Application to EACBs

With some modifications, the theory in the preceding subsection can easily be applied to ABCs for FVTD, and thus we can formulate EABCs for FVTD. In their derivation, however, careful attention should be paid to the following two points. One is the fact that there exists half time-step difference between electric and magnetic fields in the time domain, that is, n and $n' = n - 1/2$, as Yee [1] first proposed. The other is the fact that there exists one space-step difference between electric and magnetic fields for FVTD [8], contrary to the half space-step difference of FDTD. Final formulations are summarized in the following way.

First we introduce the solutions to the advection equations at $i = \pm N_x$ as follows:

$$\begin{aligned} E_y^{n*}(\pm N_x, j, k) &= E_y^n(\pm N_x, j, k) \\ &+ \frac{1}{2}\Delta_{adv}^{(\pm x)}(c\Delta t; 0, 2, 4)E_y^n(\pm N_x, j, k) \pm \frac{1}{2}\Delta_{adv}^{(\pm x)}(c\Delta t/2; 0, 1, 3)\tilde{H}_z^{n'}(\pm N_x, j, k) \\ E_z^{n*}(\pm N_x, j, k) &= E_z^n(\pm N_x, j, k) \\ &+ \frac{1}{2}\Delta_{adv}^{(\pm x)}(c\Delta t; 0, 2, 4)E_z^n(\pm N_x, j, k) \mp \frac{1}{2}\Delta_{adv}^{(\pm x)}(c\Delta t/2; 0, 1, 3)\tilde{H}_y^{n'}(\pm N_x, j, k) \end{aligned} \tag{13}$$

where $n' = n + 1/2$. Next we introduce the approximate solutions to the non-advection equations. Then, adding non-advection solutions to the former advection ones, we obtain the EABCs at $i = \pm N_x$ as follows:

$$\begin{aligned} E_y^{n+1}(\pm N_x, j, k) &= E_y^{n*}(\pm N_x, j, k) + \Delta_{non}^{(z)}(c\Delta t/2)\tilde{H}_x^{n'}(\pm N_x, j, k) \\ E_z^{n+1}(\pm N_x, j, k) &= E_z^{n*}(\pm N_x, j, k) - \Delta_{non}^{(y)}(c\Delta t/2)\tilde{H}_x^{n'}(\pm N_x, j, k) \end{aligned} \tag{14}$$

It should be noted that the propagating distance $c\Delta t/2$ instead of $c\Delta t$ is used for the magnetic field components, since there exists half time difference $\Delta t/2$ between electric and magnetic fields. It should also be noted that the sampling points of electric and magnetic fields are different each other, since there exists one space difference Δx between them. Thus we can extrapolate the boundary fields E_y^{n+1} and E_z^{n+1} as shown in Eq.(14) in terms of the fields in the computational domain in a self-consistent way.

The same discussion as in the case of x-direction can be applied to the present case where the boundaries exist at $j = \pm N_y$. The solutions to the advection equations in y-direction are expressed as follows:

$$\begin{aligned} E_z^{n*}(i, \pm N_y, k) &= E_z^n(i, \pm N_y, k) \\ &+ \frac{1}{2}\Delta_{adv}^{(\pm y)}(c\Delta t; 0, 2, 4)E_z^n(i, \pm N_y, k) \pm \frac{1}{2}\Delta_{adv}^{(\pm y)}(c\Delta t/2; 0, 1, 3)\tilde{H}_x^{n'}(i, \pm N_y, k)] \\ E_x^{n*}(i, \pm N_y, k) &= E_x^n(i, \pm N_y, k) \\ &+ \frac{1}{2}\Delta_{adv}^{(\pm y)}(c\Delta t; 0, 2, 4)E_x^n(i, \pm N_y, k) \mp \frac{1}{2}\Delta_{adv}^{(\pm y)}(c\Delta t/2; 0.1.3)\tilde{H}_z^{n'}(i, \pm N_y, k) \end{aligned} \tag{15}$$

And the EABCs at $j = \pm N_y$ are expressed as follows:

$$\begin{aligned} E_z^{n+1}(i, \pm N_y, k) &= E_z^{n*}(i, \pm N_y, k) + \Delta_{non}^{(x)}(c\Delta t/2)\tilde{H}_y^{n'}(i, \pm N_y, k) \\ E_x^{n+1}(i, \pm N_y, k) &= E_x^{n*}(i, \pm N_y, k) - \Delta_{non}^{(z)}(c\Delta t/2)\tilde{H}_y^{n'}(i, \pm N_y, k) \end{aligned} \tag{16}$$

Since the above discussions can be applied to the advections in z-direction with boundaries at $j = \pm N_z$, we obtain the solutions to the advection equations as follows:

$$\begin{aligned} E_x^{n*}(i, j, \pm N_z) &= E_x^n(i, j, \pm N_z) \\ &+ \frac{1}{2}\Delta_{adv}^{(\pm z)}(c\Delta t; 0, 2, 4)E_x^n(i, j, \pm N_z) \pm \frac{1}{2}\Delta_{adv}^{(\pm z)}(c\Delta t/2; 0, 1, 3)\tilde{H}_y^{n'}(i, j, \pm N_z)] \\ E_y^{n*}(i, j, \pm N_z) &= E_y^n(i, j, \pm N_z) \\ &+ \frac{1}{2}\Delta_{adv}^{(\pm z)}(c\Delta t; 0, 2, 4)E_y^n(i, j, \pm N_z) \mp \frac{1}{2}\Delta_{adv}^{(\pm z)}(c\Delta t/2; 0, 1, 3)\tilde{H}_x^{n'}(i, j, \pm N_z)] \end{aligned} \tag{17}$$

And the EABCs at $j = \pm N_z$ are given by the following equations:

$$\begin{aligned} E_x^{n+1}(i, j, \pm N_z) &= E_x^{n*}(i, j, \pm N_z) + \Delta_{non}^{(y)}(c\Delta t/2)\tilde{H}_z^{n'}(i, j, \pm N_z) \\ E_y^{n+1}(i, j, \pm N_z) &= E_y^{n*}(i, j, \pm N_z) - \Delta_{non}^{(x)}(c\Delta t/2)\tilde{H}_z^{n'}(i, j, \pm N_z) \end{aligned} \tag{18}$$

3. Numerical Examples

Fig. 1 shows global errors [dB] versus computation times n for 1D FVTD using Mur's ABC, PML and EABC. The errors are computed by comparing FVTD solutions of small number of cells, $N_x = 200$, with those of large number of cells, $Nx = 1200$. A source in the free space is assumed to be a continuous sinusoidal wave excited at the center of the FVTD cells with cell size of $\Delta x = \lambda/20$. The errors of EABC and Mur's ABC at $n = 400$ are -54.6 [dB] and -34.3 [dB], respectively, while that of PML with 21 absorbing cells is -73.2 [dB]. It is demonstrated that the accuracy of the present method is better than that of Mur's ABC but poorer than that of PML in case of 1D FVTD.

Fig. 2 shows global errors [dB] versus computation times n for 2D FVTD using Mur's ABC, PML and EABC. The errors are computed by comparing FVTD solutions of small number of cells, $N_x = N_y = 200$, with those of large number of cells, $Nx = N_y = 1200$. A source in the free space is assumed to be a continuous sinusoidal wave excited at the center of the FVTD cells with cell size of $\Delta x = \Delta y = \lambda/20$. The errors of EABC and Mur's ABC at $n = 600$ are -39.9 [dB] and -19.7 [dB], respectively, while that of PML with 21 absorbing layers is -61.8 [dB]. It is demonstrated that just as 1D case, the accuracy of the present method is better than that of Mur's ABC but poorer than that of PML in case of 2D FVTD.

Fig. 3 shows global errors [dB] versus computation times n for 3D FVTD using Mur's ABC and EABC. The errors are computed by comparing FVTD solutions of small number of cells, $N_x = N_y = N_z = 50$, with those of large number of cells, $Nx = N_y = N_y = 200$. A source in the free space is assumed to be a continuous sinusoidal wave excited at the center of the FVTD cells with cell size of $\Delta x = \Delta y = \Delta z = \lambda/20$. The curves exhibit ripple characteristics, and the errors of EABC and Mur's ABC at $n = 200$ are -19.1 [dB] and -18.9 [dB], respectively. It is shown that the accuracy of the present method is just as the same order of that of Mur's ABC.

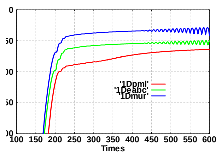

Figure 1: Errors of 1D FVTD.

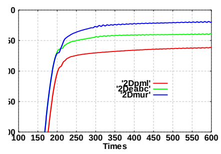

Figure 2: Errors of 2D FVTD.

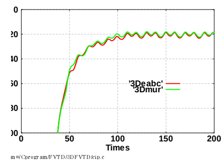

Figure 3: Errors of 3D FVTD.

4. Conclusion

In this paper, we have proposed the EABCs for FVTD method by extrapolating the electric fields at the boundaries of computational region by employing the theory of CIP. In the present formulations, we have utilized three point Lagrange's interpolation for evaluating advection terms, and we have adopted the difference method for computing non-advection terms. The advantage of the present method is that it requires almost no extra computer memories for EABCs just as Mur's ABC. Numerical calculations were carried out for checking the accuracy of EABC in comparison with Mur's ABC or PML. In cases of 1D and 2D, it has been demonstrated that the accuracy of the present method is about 20 [dB] better than that of Mur's ABC but about 20 [dB] poorer than PML.

In case of 3D, however, the proposed EABC shows almost the same order of accuracy as Mur's ABC. Improvement of the present EABCs deserve as a future investigation.

References

[1] K.S. Yee, "Numerical solution of initial value problems involving Maxwell's equation in isotropic media", *IEEE Trans. Antennas Propagat.* , Vol. AP-4, No. 3, 1966-05, pp. 302-307.

[2] A. Taflove, "Computational Electrodynamics - The Finite Difference Time-Domain Method -", *Artech House, Inc.* , Chapter 7, 1995, pp. 145-202.

[3] G. Mur, "Absorbing boundary conditions for the finite-difference approximation of the time-domain electromagnetic-field equations," *IEEE Trans. Electormagn. Compat.* , Vol. EMC-23, No. 4, 1981-11, pp. 377-382.

[4] J.P. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *J. Comp. Phys.*, Vol. 114, No. 2, 1994-10, pp. 185-200.

[5] K. Uchida, X.F. Li, H. Maeda and K.Y. Yoon, "On Extrapolated Absorbing Boundary Condition for FVTD Method", *Proceedings of ISAP2005*, 2005-08, pp. 197-200.

[6] K. Uchida, X.F. Li, H. Maeda and K.Y. Yoon, "On Extrapolated Absorbing Boundary Condition Based on CIP Method", *Proceedings of ISAP2007*, SB2, 2007-11, pp. 1-4.

[7] T. Yabe, T. Utsumi and Y. Ogata " CIP Method", *Morikita Shuppan Co., Ltd.*, Chapter 3, 2005, pp. 54-82.

[8] K. Uchida, T. Matsunaga, T. Noda and K.K. Han, "FVTD algorithm and its application procedure", *Res. Bull. Fukuoka Inst. Tech.*, Vol. 29, No. 1, 1996-10, pp. 121-130.