2E1-3

A Low Cost Absorbing Boundary Condition for FDTD Calculations Based on a Nonstandard Finite Difference Model

#James B. Cole [1], Saswatee Banerjee [2]
[1] Department of Systems Engineering, University of Tsukuba
Tennodai 1-1-1, Tsukuba, Ibaraki 305-8573, Japan, cole@is.tsukuba.ac.jp
[2] Department of Applied Physics, University of Tsukuba
Tennodai 1-1-1, Tsukuba, Ibaraki 305-8573, Japan, saswateeb@yahoo.co.in

1. Introduction

The computational domain of the FDTD algorithm must be terminated with an absorbing boundary condition (ABC). The Perfectly Matched Layer (PML) [1] is an excellent ABC, but it is complicated and costly. At least 8 layers are needed to give satisfactory absorption. In a 100^3 space only 84^3 or 59% of the grid points are usable. The second-order Mur [2] ABC requires just 2 layers, but its absorption is inadequate for some problems. In this paper we introduce an improved version of the second-order Mur ABC based on a nonstandard finite difference (NSFD) model [3] which has the same low computational cost but with much better absorption.

2. Nonstandard Finite Difference Models

Let us first derive a simple NSFD model for the one-dimensional wave equation,

$$\left(\partial_t^2 - v^2 \partial_x^2\right)\psi(x,t) = 0. \qquad (2.1)$$

Defining d_x by $d_x f(x) = f\left(x + h/2\right) - f\left(x - h/2\right)$, the central finite difference (FD) approximation to the first derivative is $f'(x) \cong d_x f(x)/h$, and the FD approximation for f'' is $f''(x) \cong d_x^2 f(x)/h^2$, where $d_x^2 f(x) = f(x+h) + f(x-h) - 2f(x)$. Replacing derivatives in (2.1) by FD approximations we obtain the standard finite difference (SFD) model,

$$\left(d_t^2 - u^2 d_x^2\right)\psi(x,t) = 0, \qquad (2.2)$$

where $u = v\Delta t/h$. Inserting $\varphi_0 = e^{i(kx - \omega t)}$ into (2.2), we find $\left(d_t^2 - u^2 d_x^2\right)\varphi_0(x,t) = \varepsilon \neq 0$. Thus φ_0 is not a solution to (2.2). The standard (S)-FDTD algorithm derived from (2.2) has error $\varepsilon \sim (h/\lambda)^2$, where $\lambda = 2\pi/k$. Regarding u in (2.2) as a free parameter, ε can be made to vanish by taking $u = u_0 = \sin(\omega\Delta t/2)/\sin(kh/2)$. This is an example of a NSFD model.

$$\left(d_t^2 - u_0^2 d_x^2\right)\psi(x,t) = 0. \qquad (2.3)$$

φ_0 is a solution to both (2.1) and (2.3), so an exact FDTD algorithm can be derived from (2.3) to solve (2.1). Exact NSFD models cannot always be found, but FDTD algorithms based on NSFD models can be much more accurate than ordinary FDTD algorithms.

3. Second-Order Mur Absorbing Boundary Condition

The two-dimensional wave equation, $\left(\partial_t^2 - v^2\partial_x^2 - v^2\partial_y^2\right)\psi(\boldsymbol{x},t) = 0$ can be factored into $\left(P + v\partial_x\right)\left(P - v\partial_x\right)\psi\left(\boldsymbol{x},t\right) = 0$, where $P = \sqrt{\partial_t^2 - v\partial_y^2}$, and $\boldsymbol{x} = (x, y)$. Thus we obtain

ISBN: 978-4-88552-223-9 C3055©IEICE

$$\left(P \pm v\partial_x\right)\psi(\boldsymbol{x},t) = 0\,, \tag{3.1}$$

the Engquist–Majda (EM) [4] one-way wave equations. Solutions are $\varphi_{\pm}(\boldsymbol{x},t) = f(\hat{\boldsymbol{k}}\cdot\boldsymbol{x} \mp vt)$, where f is an arbitrary function, and $\hat{\boldsymbol{k}} = \left(\cos\theta, \sin\theta\right)$. $\varphi_{\pm}$ propagates along the $\pm x$-direction at angle θ. Writing $P^2 = \partial_t^2\left(1 - v^2\partial_y^2 / \partial_t^2\right)$, expanding P in a Taylor series, and retaining the first two terms gives $P \cong \partial_t - \frac{1}{2}v^2\,\partial_y^2 / \partial_t$. Inserting into (3.1) and multiplying by ∂_t, we obtain the second-order EM equations,

$$\left(\partial_t^2 \pm v\partial_x\partial_t - \frac{1}{2}v^2\partial_y^2\right)\psi(\boldsymbol{x},t) = 0\,. \tag{3.2}$$

To use (3.2) FDTD calculations, we need a FD model. Taking $\Delta x = \Delta y = h$, and inserting FD approximations for the derivatives we obtain the SFD model of (3.2),

$$\left(d_t^2 \pm \frac{1}{2}\frac{v\Delta t}{h}d_x d_t' - \frac{1}{2}\frac{v^2\Delta t^2}{h^2}d_y^2\right)\psi(\boldsymbol{x},t) = 0\,, \tag{3.3}$$

where d_t' is defined by $d_t' f(t) = f(t+\Delta t) - f(t-\Delta t)$. Since ψ is sampled at $t = 0, \Delta t, 2\Delta t, \cdots$, we use $\partial_t\psi \cong d_t'\psi / 2\Delta t$. On the computational domain $\left(x = 0, h, \cdots N_x h\right) \times \left(y = 0, h, \cdots N_y h\right)$ we set the grid points on the outer boundary to zero, i.e. $\psi(0,y) = \psi(N_x, y) = \psi(x,0) = \psi(x, N_y) = 0$, and evaluate (3.3) at points $x = b$, where $b = h$ and $b = \left(N_x - 1\right)h$. Let i be one grid spacing inside the boundary. Thus $i = b \pm h$, at $b = h$ and $b = \left(N_x - 1\right)h$, respectively. The midpoint between b and i is $m = \left(b + i\right)/2$. Writing $\psi\left(x,y,t\right) = \psi_{x,y}^{t}$ and $\psi\left(x,y,t \pm \Delta t\right) = \psi_{x,y}^{t\pm1}$, we evaluate (3.3) at $x = m$, with the approximation $\psi_{m,y}^{t} \cong \left(\psi_{b,y}^{t} + \psi_{i,y}^{t}\right)/2$ and obtain

$$d_t^2\left(\psi_{b,y}^{t} + \psi_{i,y}^{t}\right) + \bar{v}\left[\left(\psi_{b,y}^{t+1} - \psi_{i,y}^{t+1}\right) - \left(\psi_{b,y}^{t-1} - \psi_{i,y}^{t-1}\right)\right] - \frac{1}{2}\bar{v}^2 d_y^2\left(\psi_{b,y}^{t} + \psi_{i,y}^{t}\right) = 0\,, \tag{3.4}$$

where $\bar{v} = v\Delta t/h$. The ($\pm$) in (3.3) becomes a (+) sign in (3.4) because $\left(\psi_{b,y}^{t\pm1} - \psi_{i,y}^{t\pm1}\right)$ has opposite signs on at $b = h$ and $b = \left(N_x - 1\right)h$. Expanding $d_t^2\psi_{b,y}^{t}$, and solving for $\psi_{b,y}^{t+1}$ yields,

$$\psi_{b,y}^{t+1} = \psi_{b,y}^{t} + \left(\psi_{i,y}^{t} - \psi_{i,y}^{t-1}\right) + \left(\frac{1-\bar{v}}{1+\bar{v}}\right)\left[\left(\psi_{b,y}^{t} - \psi_{b,y}^{t-1}\right) - \left(\psi_{i,y}^{t+1} - \psi_{i,y}^{t}\right)\right] + \frac{1}{2}\left(\frac{\bar{v}^2}{1+\bar{v}}\right)d_y^2\left(\psi_{b,y}^{t} + \psi_{i,y}^{t}\right). \tag{3.5}$$

A similar form for the $\pm y$ directions can be derived. Algorithm (3.5) is the standard (S) Mur ABC. Let us now evaluate the performance of (3.5).

Define the left side of (3.4) to be $A_{\text{SFD}}\psi$, where A_{SFD} is a S-Mur annihilation operator. We now evaluate the effect A_{SFD} of an incident wave $\varphi_{\pm} = e^{i(\boldsymbol{k}\cdot\boldsymbol{x} \mp \omega t)}$, where $\boldsymbol{k} = k\hat{\boldsymbol{k}} = \left(k_x, k_y\right)$, and $\omega = vk$. $A_{\text{SFD}}\varphi_{\pm} = \varepsilon_{\text{SFD}}\varphi_{\pm}$, where ε_{SFD} is the annihilation error. Writing $\tilde{\varepsilon}_{\text{SFD}} = \varepsilon_{\text{SFD}} / 8\sin^2\left(\bar{\varpi}/2\right)$, $\omega\Delta t = \bar{\varpi}$, $kh = \bar{k}$, $k_{x,y}h = \bar{k}_{x,y}$, and using the identities, $\psi_{b,y}^{t} + \psi_{i,y}^{t} = 2\cos\left(k_x h/2\right)\psi_{m,y}^{t}$, $\psi_{b,y}^{t} - \psi_{i,y}^{t} = 2i\sin\left(k_x h/2\right)\psi_{m,y}^{t}$, and $d_y^2\psi_{x,y}^{t} / \psi_{x,y}^{t} = -4\sin^2\left(k_y h/2\right)$, we find

$$\tilde{\varepsilon}_{\text{SFD}}(\theta) = -\cos\left(\bar{k}_x/2\right) + \bar{v}\,\frac{\sin\left(\bar{k}_x/2\right)}{\tan\left(\bar{\varpi}/2\right)} + \frac{1}{2}\bar{v}^2\,\frac{\sin^2\left(\bar{k}_y/2\right)}{\sin^2\left(\bar{\varpi}/2\right)}\cos\left(\bar{k}_x/2\right). \tag{3.6}$$

Expanding in a Taylor series about $\sin\theta = 0$ gives

$$\tilde{\varepsilon}_{\text{SFD}}(\theta)=\left[-\cos\left(\bar{k}/2\right)+\frac{\bar{v}\sin\left(\bar{k}/2\right)}{\tan\left(\bar{\omega}/2\right)}\right]+ \tag{3.7}$$
$$\left[-\frac{1}{4}\bar{k}\sin\left(\bar{k}/2\right)-\frac{1}{4}\bar{v}\bar{k}\frac{\cos\left(\bar{k}/2\right)}{\tan\left(\bar{\omega}/2\right)}+\frac{1}{8}\bar{v}^2\bar{k}^2\frac{\cos\left(\bar{k}/2\right)}{\sin^2\left(\bar{\omega}/2\right)}\right]\sin^2(\theta)+\cdots.$$

Expanding the terms of (3.6) in powers of $\bar{k}$, we find that the reflectivity of the S-Mur ABC is proportional to

$$\tilde{\varepsilon}_{\text{SFD}}(\theta)=\frac{1}{12}\left(1-\bar{v}^2\right)k^2h^2+\left[\frac{1}{12}\bar{v}^2-\frac{1}{8}\right]k^2h^2\sin^2\theta+\cdots. \tag{3.8}$$

4. Nonstandard Second-Order Mur Absorbing Boundary Condition

In the NSFD model of (3.2) we replace $\bar{v}$ and $\bar{v}^2$ in (3.4) by the free parameters in u_1 and u_2^2, respectively. The NSFD annihilation operator, A_{NSFD}, is given by

$$A_{\text{NSFD}}\psi=d_t^2\left(\psi_{b,y}^t+\psi_{i,y}^t\right)+u_1\left[\left(\psi_{b,y}^{t+1}-\psi_{i,y}^{t+1}\right)-\left(\psi_{b,y}^{t-1}-\psi_{i,y}^{t-1}\right)\right]-\frac{1}{2}u_2^2d_y^2\left(\psi_{b,y}^t+\psi_{i,y}^t\right). \tag{4.1}$$

Evaluating $\varepsilon_{\text{NSFD}}=A_{\text{NSFD}}\psi_\pm/\psi_\pm$, we seek values of u_1 and u_2^2 that minimize $\tilde{\varepsilon}_{\text{NSFD}}$ $=\varepsilon_{\text{NSFD}}/8\sin^2(\bar{\omega}/2)$. Defining $u_1=w_1\tan\left(\bar{\omega}/2\right)$, and $u_2=w_2\sin\left(\bar{\omega}/2\right)$, and putting $\bar{v}\to u_1$, and $\bar{v}^2\to u_2^2$ in (3.6) we obtain

$$\tilde{\varepsilon}_{\text{NSFD}}(\theta)=-\cos\left(\bar{k}_x/2\right)+w_1\sin\left(\bar{k}_x/2\right)+\frac{1}{2}w_2^2\sin^2\left(\bar{k}_y/2\right)\cos\left(\bar{k}_x/2\right). \tag{4.2}$$

Taking $w_1=\cot\left(\bar{k}/2\right)$ ensures that $\tilde{\varepsilon}_{\text{NSFD}}(0)=0$. Inserting this value of w_1 into (4.2) yields

$$\tilde{\varepsilon}_{\text{NSFD}}(\theta)=\delta_0+\frac{1}{2}w_2^2\delta_2, \tag{4.3}$$

where $\delta_0=-\cos\left(\bar{k}_x/2\right)+\cot\left(\bar{k}/2\right)\sin\left(\bar{k}_x/2\right)$, and $\delta_2=\sin^2\left(\bar{k}_y/2\right)\cos\left(\bar{k}_x/2\right)$. It remains to choose w_2^2. Expanding (3.3) about $\sin\theta=0$ we find

$$\tilde{\varepsilon}_{\text{NSFD}}(\theta)=\left[-\frac{1}{4}\frac{\bar{k}}{\sin\left(\bar{k}/2\right)}+\frac{1}{8}w_2^2\bar{k}^2\cos\left(\bar{k}/2\right)\right]\sin^2\theta+\cdots. \tag{4.4}$$

Choosing $w_2^2=4/\bar{k}\sin\bar{k}$ cancels the $\sin^2\theta$-term in (3.4), but we have found that a better overall reduction of $\left|\tilde{\varepsilon}_{\text{NSFD}}\right|$ can be achieved by choosing w_2^2 such that the $\sin^2\theta$-term partially cancels the higher order terms. We chose w_2^2 such that the mean value of $\tilde{\varepsilon}_{\text{NSFD}}$ vanishes on the range $0\le\theta\le\varphi=\pi/6$. One would like to choose $\varphi=\pi/2$, but the larger φ, the larger the mean value of $\left|\tilde{\varepsilon}_{\text{NSFD}}(\theta)\right|$. We find that $w_2^2=\left(21/5\right)/\bar{k}\sin\bar{k}$ is an optimal choice. We thus obtain

$$u_1=\frac{\tan\left(\omega\Delta t/2\right)}{\tan\left(kh/2\right)},\quad u_2^2=\frac{21}{5}\frac{\sin^2\left(\omega\Delta t/2\right)}{\left(kh\right)\sin\left(kh\right)}. \tag{4.5}$$

The NS (nonstandard)-Mur ABC is thus obtained from the S-Mur ABC by putting $\bar{v}\to u_1$, and $\bar{v}^2\to u_2^2$ in (3.5).

$$\psi_{b,y}^{t+1}=\psi_{b,y}^t+\left(\psi_{i,y}^t-\psi_{i,y}^{t-1}\right)+u_1\left[\left(\psi_{b,y}^t-\psi_{b,y}^{t-1}\right)-\left(\psi_{i,y}^{t+1}-\psi_{i,y}^t\right)\right]+\frac{1}{2}u_2^2d_y^2\left(\psi_{b,y}^t+\psi_{i,y}^t\right). \tag{4.6}$$

We have shown that the algorithm in this form is numerically stable, but it must be carefully implemented. The choices $b = h, (N_x - 1)h$ suppress the propagation of unabsorbed corner fields into the computational domain. Although $\psi_{b,b}^{t+1}$ can be evaluated with either the ABC for $\psi_{b,y}^{t+1}$ or $\psi_{y,b}^{t+1}$, we have found that the result is insensitive to which one is used.

5. Results and Conclusions

In Fig. 1 we compare the performance of the NS-Mur ABC with the S-Mur ABC for a normally incident gaussian pulse as shown. Even where the pulse rises and falls, the NS-Mur ABC reduces the reflection by a factor of 10^{-1} compared with the S-Mur ABC, and elsewhere the reduction is more than 10^{-3}. Fig. 2 shows the same comparison for a pulse incident at 30° to the normal of the boundary. Using the NS-Mur ABC reduces the reflected intensity by factor of $1/37$.

The NS-Mur ABC is no more costly than the Mur ABC, but it gives much better absorption. When near perfect absorption is required at high incidence angles there is no choice but to use PML with its high costs, but in many problems the NS-Mur ABC is an adequate low-cost alternative.

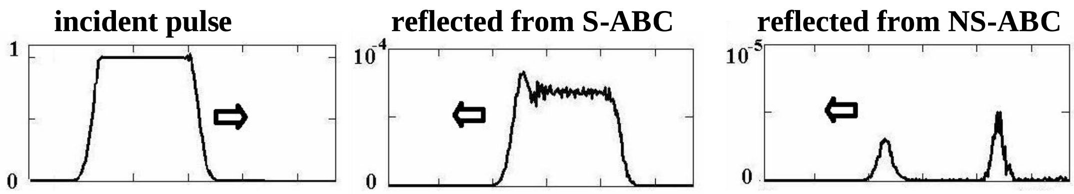

Figure 1. Comparison of the S-Mur ABC and NS-Mur ABC for normal incidence

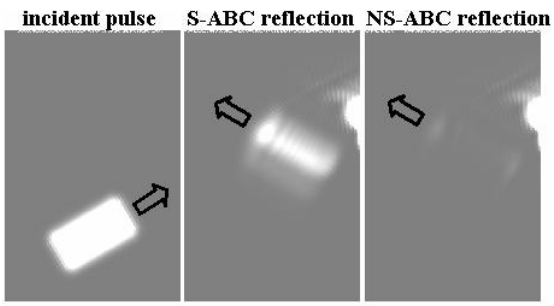

Figure 2. Comparison of the S-Mur ABC and NS-Mur ABC for incidence at 30° to normal.

References

[1] A. Taflove, S.C. Hagness, *Computational Electrodynamics*," 2nd Edition, Artech House, Norwood, MA, USA, pp 172-236, 2000.
[2] G. Mur, "Absorbing boundary conditions for the finite-difference approximation of the time domain electromagnetic field equations," IEEE Transactions on Electromagnetic Compatibility, vol. 23, pp. 377-382, 1981.
[3] R. E. Mickens, "Nonstandard Finite Difference Models of Differential Equations," World Scientific, Singapore, 1994.
[4] B. Engquist, A. Majda, "Absorbing boundary conditions for the numerical simulation of waves," Mathematics of Computation, vol. 31, pp. 629-651, 1977.
[5] J. B. Cole, "Application of Nonstandard Finite Differences to Solve the Wave Equation and Maxwell's Equations," in chap. 3 of R. E. Mickens, editor, "Applications of Nonstandard Finite Difference Schemes," World Scientific, Singapore, 2000.