

## AN EIGENVALUE EQUATION FOR THE CHARACTERISTIC POLARIZATION STATES IN THE CROSS-POLARIZED RADAR CHANNEL

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### 1. Introduction

As regards the characteristic polarization states of a radar target for the completely polarized wave case, Boerner et. al. [1], [2] have already derived eight characteristic polarization states based on the polarization transformation ratio, for which a radar receives optimum power. These states are two co-polarization maximums (CO-POL Maxs), two co-polarization nulls (CO-POL Nulls), two cross-polarization maximums (X-POL Maxs), two cross-polarization saddles (X-POL Saddles), and two cross-polarization nulls (X-POL Mins). Since the pair of X-POL Nulls and CO-POL Maxs is identical, there exists a total of eight physical characteristic polarization states.

The purpose of this paper is to present an alternative method for deriving the characteristic polarization states in the cross-polarized radar channel for the monostatic reciprocal case based on a Stokes vector formulation [3] - [5]. The Stokes vector formulation has an advantage in its applicability for finding solutions for both completely polarized wave and partially polarized wave cases. In the following, we show that the optimization procedure to the cross-channel power for the coherent case leads to an eigenvalue equation which explains the characteristic polarization state properties mathematically and physically.

### 2. Cross-polarized Channel Power

Consider the case for which a monostatic radar transmits a completely polarized (coherent) wave and receives a coherent scattered wave from a target. Assuming that the transmitted wave has unit magnitude, the wave can be expressed in terms of Stokes vector as follows

$$\mathbf{g}_{tr}^T = ( 1, x_1, x_2, x_3 ), \quad (1)$$

where T denotes transpose, and  $x_i$  ( $i=1, 2, 3$ ) is the component of Stokes vector  $\mathbf{g}_{tr}$  which constitutes sub-Stokes vector  $\tilde{\mathbf{X}}$ .

$$\tilde{\mathbf{X}}^T = ( x_1, x_2, x_3 ). \quad (2)$$

The radar is assumed to have two polarimetric receiving channels; the first channel has a co-polarized receiving antenna whose polarization state is the same as that of the transmitting antenna, the other has a cross-polarized antenna whose polarization state is orthogonal. The channel power depends on the transmitting polarization state and target scattering property such as shape, orientation, size, material, etc. Since the scattering property of a target cannot be controlled, we obtain the target information by changing polarization state of transmitting wave in a polarization agile radar. The problem here is to find polarization states for which the cross-polarized channel power is optimal for a given target. If we concentrate on polarimetric information excluding amplitude dependency due to path length in a scattered wave, the power in the cross-polarized radar channel in terms

of transmitting Stokes vector is expressed [3] as follows

$$P_x = \frac{1}{2} \mathbf{g}_{tr}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [\mathbf{M}] \mathbf{g}_{tr} = \frac{1}{2} ( -\tilde{\mathbf{X}}^T [\mathbf{N}] \tilde{\mathbf{X}} + m_{00} ), \quad (3)$$

where  $[\mathbf{M}]$  is defined as Mueller matrix representing scattering property of target

$$[\mathbf{M}] = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix}, \quad (4)$$

and we define

$$[\mathbf{N}] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ -m_{31} & -m_{32} & -m_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & -m_{33} \end{bmatrix}. \quad (5)$$

It should be noted that the matrix  $[\mathbf{N}]$  is symmetric for the monostatic reciprocal case.

### 3. Eigenvalue Problem and Characteristic Polarization States

We optimize the X-POL power (3) subject to the constraint

$$\Phi = \sqrt{x_1^2 + x_2^2 + x_3^2} - 1 = 0. \quad (6)$$

This constraint is due to an assumption that the transmitted wave is coherent. The optimization procedure employing Lagrangian method with multiplier  $\mu$

$$\frac{\partial P_x}{\partial x_i} - \mu \frac{\partial \Phi}{\partial x_i} = 0 \quad (i = 1, 2, 3) \quad (7)$$

leads to the following matrix equation

$$- \begin{bmatrix} m_{11} + \mu & m_{12} & m_{13} \\ m_{12} & m_{22} + \mu & m_{23} \\ m_{13} & m_{23} & \mu - m_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (8)$$

This equation reduces to an eigenvalue equation of general form.

$$[-\mathbf{N}] \tilde{\mathbf{X}} = \lambda \tilde{\mathbf{X}}, \quad (9)$$

where  $\lambda$  is the eigenvalue of  $\tilde{\mathbf{X}}$  and is equal to  $\mu$  in this case. Mathematically,  $P_x$  in eq(3) is essentially of Hermitian form, hence the optimization of  $P_x$  leads to the eigenvalue equation (9). Since  $[-\mathbf{N}]$  is also real and symmetric, we find from mathematical point of view that

1. This  $3 \times 3$  matrix equation has three real eigenvalues,  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$

( $\lambda_1 > \lambda_2 \geq \lambda_3$ ) including degeneracy of  $\lambda_2 = \lambda_3$ .

2. The eigenvectors corresponding to the eigenvalues are orthogonal to each other.

The solutions to eq(9) provide stationary points in  $P_x$  and characteristic polarization states for a given target  $[\mathbf{M}]$ . From the first property and from the fact that the matrix  $[-\mathbf{N}]$  is of Hermitian form associated with physical power, the largest eigenvalue  $\lambda_1$  gives the largest power (X-POL Max), the smallest eigenvalue  $\lambda_3$  gives the minimal (X-POL Min), while the intermediate eigenvalue  $\lambda_2$  gives the second max or the second min (X-POL

Saddle) when  $\lambda_2 \neq \lambda_3$ . For the case of  $\lambda_2 = \lambda_3$ , the X-POL Saddle points vanish and hence the corresponding power does not exist. The power in the characteristic polarization state associated with the eigenvalue is given by

$$P_x = \frac{1}{2} (\lambda_i + m_{00}) \quad (10)$$

From the second property, the eigenvectors are orthogonal to each other, which, in turn, implies these solution sub-Stokes vectors are orthogonal. Since the basis vectors of sub-Stokes vector  $\tilde{\mathbf{X}}$  constitute three rectangular coordinate axes of the Poincaré sphere, solution vectors constitutes a new frame of rectangular coordinate in the Poincaré sphere [2] due to this spatial property. Hence, this spatial orthogonality on the Poincaré sphere always applies to the property of characteristic polarization states in the cross-polarized radar channel.

For a given eigenvalue, say  $\lambda_1$ , we obtain two solution vectors under the condition (6), that is, if  $\mathbf{X}_1^T = (x_1, x_2, x_3)$  is a solution vector, then  $\mathbf{X}_2^T = (-x_1, -x_2, -x_3)$  also becomes the solution vector. The condition  $\mathbf{X}_1^T \mathbf{X}_2 = -1$  is the polarimetric orthogonality condition for two polarization states. Since the tip of a solution vector on the Poincaré sphere surface represents a characteristic polarization state, the tips of these two solution vectors must locate on the anti-podal points on the Poincaré sphere representing orthogonal polarization states to each other. Even though the polarization states are orthogonal, they produce the same power because the pair solution vectors are determined from the same eigenvalue.

## 5. Numerical Examples

If a Mueller matrix [M] is given as

$$[M] = \begin{bmatrix} 1.0000 & 0.0762 & 0.1399 & 0.0264 \\ 0.0762 & 0.7682 & 0.3832 & -0.0615 \\ 0.1399 & 0.3832 & -0.2302 & 0.0596 \\ -0.0264 & 0.0615 & -0.0596 & -0.4619 \end{bmatrix},$$

then the eigenvalue equation becomes

$$\begin{bmatrix} -0.7682 & -0.3832 & 0.0615 \\ -0.3832 & 0.2302 & -0.0596 \\ 0.0615 & -0.0596 & -0.4619 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The eigenvalues and solution vectors are listed in Table I. The power spectrum as a function of the transmitting polarization state, tilt angle and ellipticity angle, is illustrated in Fig.1(a). One can find six stationary points which correspond to the characteristic polarization states (X-POL Maxs, X-POL Mins, and X-POL Saddles) in the cross-polarized channel. These points are displayed on the Poincaré polarization sphere in Fig.1(b). It should be noted in Fig.1(b) that each pair locates anti-podal points on the sphere and that three lines connecting each pair intersect at the origin at right angle.

## 6. Conclusion

Although these characteristic polarization states derived by this method are exactly the same as those derived by Sinclair matrix optimization method using the polarization transformation ratio [2], the formulation associated with this eigenvalue equation provides a comprehensive physical and mathematical interpretation.

## References

- [1] A.P.Agrawal and W-M.Boerner, "Redevelopment of Kennaugh's target characteristic polarization state theory using the polarization transformation ratio formalism for the coherent case," *IEEE Trans. Geosci. Remote Sensing*, vol.27, 1, pp. 2-13, 1989.
- [2] W-M.Boerner and A-Q.Xi, "The characteristic radar target polarization state theory for the coherent monostatic and reciprocal case using the generalized polarization

- transformation ratio formulation," *AEÜ*, Band 44, Heft 4, pp.273-281, 1990.
- [3] Y.Yamaguchi, M.Sengoku, T.Abe, K.Sasagawa, and W-M.Boerner, "On the characteristic polarization states of coherently reflected waves in radar polarimetry", *Technical Report of IEICE Japan*, A.P90-35, 1990.
- [4] *ibid*, "Property of characteristic polarization states in the cross polarized radar channel", *The 3rd Asia-Pacific Microwave Conf.*, Tokyo, pp.631-634, 1990.
- [5] W.-M.Boerner, W.L.Yan, A-Q.Xi, and Y.Yamaguchi, "On the basic principles of radar polarimetry: the target characteristic polarization state theory of Kennaugh, Huynen's polarization fork concept, and its extension to the partially polarized case", *Proc. of the IEEE*, vol.79, no.10, pp.1538-1550, 1991.

Table I Eigenvalues, eigenvectors, and power

eigenvalue	power	$x_1$	$x_2$	$x_3$	characteristic pol. state
$\lambda_1 = 0.3673$	0.6837	- 0.3228	0.9420	- 0.0917	Max
	0.6837	0.3228	- 0.9420	0.0917	
$\lambda_2 = -0.4654$	0.2673	0.0553	0.1155	0.9918	Saddle
	0.2673	- 0.0553	- 0.1155	- 0.9918	
$\lambda_3 = -0.9018$	0.0491	0.9448	0.3151	0.0894	Min
	0.0491	- 0.9448	- 0.3151	- 0.0894	

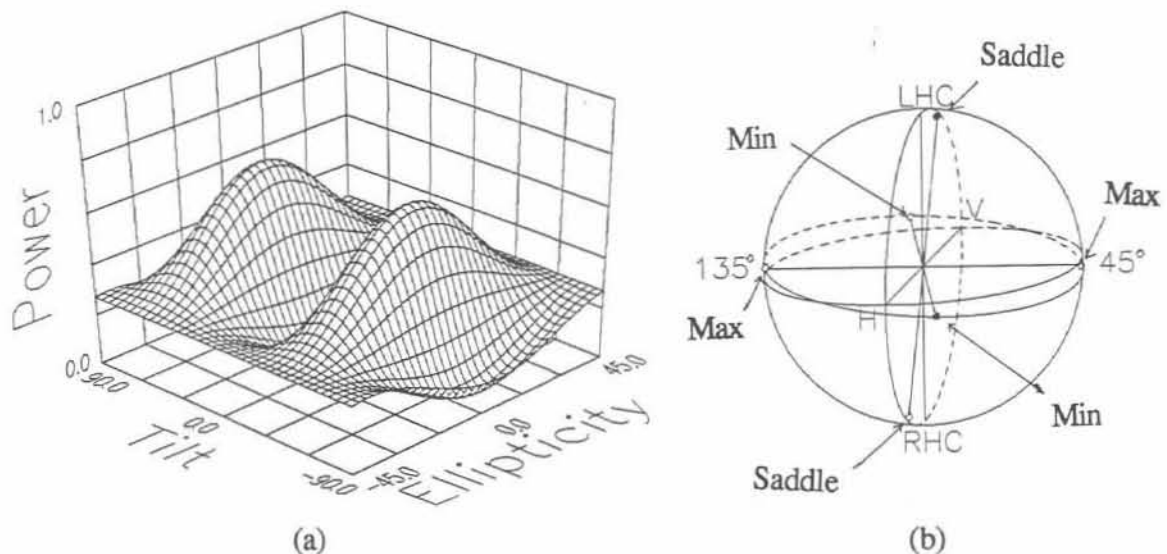


Fig.1 Example of characteristic polarization states. (a) Cross-polarized channel power as a function of transmitting polarization state, (b) Characteristic polarization states on Poincaré sphere.