POLYNOMIAL APPROXIMATIONS FOR THE ELECTRIC POLARIZABILITIES OF SOME SMALL APERTURES

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I INTRODUCTION

In several branches of electromagnetic engineering there is a need to determine the polarizabilities of apertures of various shapes. This paper is concerned with the electric polarizabilities of small apertures of the shapes shown in Figure 1, i.e. rectangle, diamond, rounded end slot and ellipse, of which only the ellipse has an exact solution. All of the shapes have a maximum length L and a maximum width W, and the width to length ratio W/L is designated α.

II RECTANGLE

Recently, Arvas and Harrington [1] have given numerical values for the electric polarizabilities of rectangular apertures of various W/L ratios, as the duals of the magnetic polarizabilities of conducting disks. Their values compare well with those calculated earlier [2,3] using a variational technique, and with Cohn's experimental values [4].

The electric polarizability of a rectangular aperture as in Figure 1(a) may be expressed as

$$Pe = R_E L^3 \tag{1}$$

in which the coefficient R_E is a function of the ratio W/L i.e.

$$R_E = f\left(\frac{W}{L}\right) \tag{2}$$

The polarizability of a square (W = L) is of particular interest because that value determines the slope of a function in addition to its magnitude as will now be shown. The electric polarizability of a rectangular aperture is independent of the choice of which side is L and which is W. Thus from (1) and (2)

$$f\left(\frac{W}{L}\right)L^3 = f\left(\frac{L}{W}\right)W^3 = \left(\frac{W}{L}\right)^3 f\left(\frac{L}{W}\right)L^3$$

or $$f(\alpha) = \alpha^3 f\left(\frac{1}{\alpha}\right) \tag{3}$$

Taking the derivative of (3) with respect to α and letting $\alpha = 1$ gives

$$f'(1) = \frac{3}{2} f(1) \tag{4}$$

Therefore a numerical value for the polarizability coefficient for a square aperture, together with the knowledge [4,5] that as

$$\alpha \to 0 \qquad f(\alpha) \to \frac{\pi}{16}\alpha^2 \tag{5}$$

gives a considerable amount of information about $f(\alpha)$. For example if $f(\alpha)$ is approximated by the polynomial

$$f(\alpha) = a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4$$

for α in the range 0 to 1, then from (5) $a = 0$, $b = 0$ and $c = \frac{\pi}{16}$. The use of (4) with f(1) taken as 0.1126 from [3] enables d and e to be determined. The resulting polynomial expression for $f(\alpha)$, which is the polarizability coefficient R_E, can then be expressed as

$$f(\alpha) = \frac{\pi}{16}\alpha^2 \{1.0 - 0.5663\alpha + 0.1398\alpha^2\} \qquad (6)$$

III DIAMOND

For the diamond shape in Figure 1(b), the electric polarizability can be expressed as

$$Pe = g(\alpha)\, L^3$$

in which the coefficient $g(\alpha)$ is a function of the ratio W/L. For the diamond, the choice of L or W for the reference direction is arbitrary, as it is for the rectangle, leading to

$$g'(1) = \frac{3}{2}\, g(1)$$

Thus if the electric polarizability of a square is known to good accuracy, two equations are available for the determination of the coefficients of a fourth power polynomial. However whereas for the rectangle the behaviour was known for $\alpha \to 0$, for the diamond shape some intuitive reasoning is necessary to ascertain the small α behaviour.

For a rectangular aperture, as the ratio W/L goes to zero,

$$Pe \to \frac{\pi}{16}\left(\frac{W}{L}\right)^2 L^3 = \frac{\pi}{16} W^2 L$$

which may be interpreted as a polarizability of $\frac{\pi}{16} W^2$ per unit length. This suggests that if the width ω of a long narrow aperture varies very slowly along the length, then the polarizability could be obtained by integrating $\frac{\pi}{16}\omega^2$ along the length of the aperture. This postulate is supported by the fact that if it is used to calculate the electric polarizability of a very long narrow ellipse (as in Fig. 1 (d) but with $W \ll L$), the result agrees with the exact solution from [6] for an ellipse of eccentricity approaching unity.

The application of this reasoning to the diamond shape gives

for $\alpha \to 0$ $\qquad g(\alpha) \to \frac{\pi}{48}\alpha^2$

If for α in the range $0 < \alpha < 1$, $g(\alpha)$ is approximated by

$$g(\alpha) = a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4$$

then $\qquad a = 0$, $b = 0$ and $c = \frac{\pi}{48}$

Also using the results for a square from [3], for a diamond with W = L the polarizability is

$$0.1126\left(\frac{L}{\sqrt{2}}\right)^3$$

leading to $g(1) = 0.0398$ and $g'(1) = 0.0597$. Then the resulting polynomial can be put in the form

$$g(\alpha) = \frac{\pi}{48}\alpha^2\{1.0 - 0.4794\alpha + 0.0876\,\alpha^2\} \qquad (7)$$

IV ROUNDED END SLOT

If for the rounded end slot shown in Figure 1(c) the electric polarizability is expressed as

$$Pe = h(\alpha)L^3$$

then $h(1) = \frac{1}{12}$ as that is the known result for a circle [6]. Also as $\alpha \to 0$, $h(\alpha)$ for a rounded end slot will approach the same value as for a rectangle. Therefore as

$$\alpha \to 0 \qquad h(\alpha) \to \frac{\pi}{16}\alpha^2$$

What is missing in this case is a direct way of obtaining $h'(1)$ as there is no equivalent of equation (3) ($\alpha>1$ has no meaning).

Consider the effect on the polarizability of a circular aperture of radius R if the radius is reduced very slightly by δ as in Figure 2(a).

$$Pe = \frac{2}{3}(R - \delta)^3 \qquad \approx \frac{2}{3}R^3 - 2R^2\delta$$

Thus for incrementally small values of δ, the change in Pe is proportional to δ, and is $\frac{R}{\pi}$ times the change in area. Also it is known that the electric polarizability is not orientation dependent. This suggests that if δ is not uniform around the boundary, provided it is very small and its variation is smooth, then the change in polarizability may be obtained by integrating δ around the circumference, i.e. from $\frac{R}{\pi}$ times the change in area. This postulate is supported by the fact that if it is applied to an elliptical aperture of very small eccentricity, by considering the ellipse to be a slight deformation of a circle, the result for the polarizability agrees with the exact solution from [6] for an ellipse of eccentricity approaching zero.

Then $h'(1)$ can be obtained by considering the limit as $\alpha \to 1$ of the rounded end slot within a circle as shown in Figure 2(b). The result is

$$h'(1) = \frac{1}{4} - \frac{1}{2\pi}$$

The solution then proceeds as for the other aperture shapes to give

$$h(\alpha) = \frac{\pi}{16}\alpha^2\{1.0 - 0.7650\alpha + 0.1894\alpha^2\} \qquad (8)$$

V ELLIPSE

There may be some interest in a fourth power polynomial expression for the electric polarizability of an elliptical aperture, either as a simpler alternative to the exact solution [6] containing an elliptic integral, or to ascertain for this case how close the polynomial approach is to the exact solution (the ellipse is the only shape for which this test can be applied). For an ellipse, the polynomial expression for the polarizability coefficient, obtained by the methods outlined above, is

$$i(\alpha) = \frac{\pi}{24}\alpha^2\{1.0 - 0.4085\alpha + 0.0451\alpha^2\} \qquad (9)$$

From the comments made in Sections III and IV, it is known that $i(\alpha)$ has the correct behaviour as $\alpha \to 0$, and has the correct magnitude and slope at $\alpha = 1$.

VI CONCLUSIONS

Polynomial approximations for the electric polarizability coefficients of some small apertures are presented in equations (6),(7),(8) and (9). When multiplied by L^3, those expressions give the electric polarizabilities of the respective apertures. Although the polynomials are not exact, all embody features which would exist in exact solutions. In the case of an ellipse where a comparison with an exact solution is possible, the average error for W/L over the range 0 to 1 is less than 2% and the maximum error is approximately 3%. In the cases of the rectangle, diamond and rounded end slot, the polarizability values calculated from the polynomials compare well with all previously published numerical and experimental data indicating accuracy sufficient for many purposes.

REFERENCES

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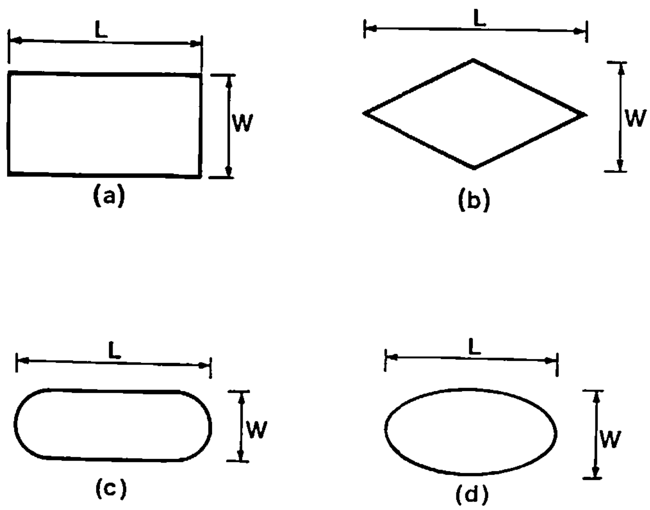

Figure 1

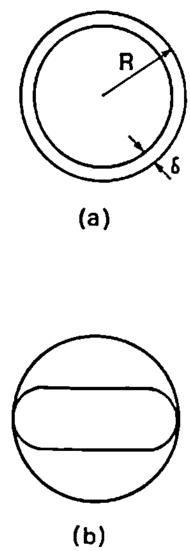

Figure 2