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# SCATTERING OF CYLINDRICAL WAVES BY A SPHERICAL OBJECT

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## SYNOPSIS

A theory is developed to establish some relations between cylindrical and spherical wave functions. Some of these results can be reduced to well-known formulae as special cases. The representations thus obtained have been utilized for the problem of scattering of cylindrical waves by a spherical object.

The problems involving scattering or diffraction of plane waves or spherical waves by cylindrical and spherical objects are well known and can be found in standard literature.[1-4] Since spherical waves can be generated by a dipole (or point source), the scattering of spherical waves by an object is a phenomenon equivalent to the radiation of a dipole in the presence of that object. Similarly, the scattering of cylindrical waves is equivalent to the radiation from a uniform line source (infinitely long) in the presence of the object under consideration. The success in obtaining rigorous solutions of such problems, namely, scattering of plane waves or spherical waves by cylindrical or spherical objects, lies in the ability to represent plane, as well as spherical waves in terms of either cylindrical or spherical wave functions. Since a cylindrical wave can easily be expressed as a superposition of cylindrical wave functions, the problem of radiation from a line source in the presence of an infinitely long cylindrical object with its axis in the same direction has also been studied successfully in literature. However, the analysis of the radiation or scattering of a line source in the presence of a spherical object encounters mathematical difficulties. This problem becomes more complicated when the source of radiation is electromagnetic in origin. The topic of the present paper addresses itself to such a study.

There are at least two methods of attacking this problem. Since the scattered fields of a line source in the presence of a sphere have vector character, one may attempt as a first method to establish a relation between the cylindrical dyadic Green's function and the spherical dyadic Green's function, both being appropriate for free space (i.e., without regard to the presence of the sphere). For the second method, one may represent the radiation from a line source in free space in terms of the radiation field of the corresponding dipole source for which the dipole field is expressed as an expansion of spherical wave functions. Although these two methods appear to be different, they produce the same results which can be used for the actual problem (i.e., when the sphere is present). This procedure enables one to represent some cylindrical functions in terms of spherical wave functions in the form of inifinite series or integrals. In establishing such representation, it will be necessary to invoke some unfamiliar orthogonality relations[5] involving spherical and cylindrical functions.

In the preceding paragraphs the general outline of the method of procedure is given. Due to lengthy and involved mathematics, it is not possible to present details here. Instead, as an example, we consider a specific problem. Namely, the radiation of an electric line source oriented parallel to the z-axis in the presence of a perfectly conducting sphere with its center at the origin. Since it is necessary to employ both the cylindrical and spherical coordinates simultaneously, we designate $(\rho, \phi, z)$ and $(r, \theta, \phi)$ to represent cylindrical and spherical coordinates. Maxwell's equations (in M.K.S. units) appropriate for this investigation may be expressed in the following manner (suppressing the assumed harmonic time dependence, $e^{-i\omega t}$):

$$\nabla \times \vec{E} = i\omega\mu_o \vec{H} \, , \tag{1a}$$

$$\nabla \times \vec{H} = -i\omega\epsilon_o \vec{E} + \frac{\hat{z}_o \mathcal{J} \delta(\rho-\rho_o)\delta(\phi-\phi_o)}{\rho} \, , \tag{1b}$$

where $\mathcal{J}$ is a constant representing the strength of the line source per unit length and $\hat{z}_o$ is the unit vector in the polar or z-direction. The radial distance (in cylindrical coordinates), $\rho_o$, is longer than the radius, a, of the sphere. The nonvanishing incident fields from the electric line source consist of $E_z^{inc}$, $H_\phi^{inc}$ and $H_\rho^{inc}$

$$E_z^{inc} = -\frac{\mathcal{J}\omega\mu_o}{4} \sum_{m=0}^{\infty} (2-\delta_{o,m}) \cos m(\phi-\phi_o) \times J_m(K\rho_<) H_m^{(1)}(K\rho_>) \, , \tag{2a}$$

where $\delta_{\ell,m}$ is the well known Kronecker delta.

$$H_\phi^{inc} = \frac{1}{i\omega\mu_o} \frac{1}{\rho} \frac{\partial}{\partial \phi} E_z^{inc} \, , \tag{2b}$$

$$H_\rho^{inc} = -\frac{1}{i\omega\mu_o} \frac{\partial}{\partial \rho} E_z^{inc} \, . \tag{2c}$$

The scattered fields from the sphere may be expressed in terms of two potentials $A_r$ and $F_r$ in spherical coordinates as

$$A_r = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \hat{B}_{m,n} r h_n^{(1)}(Kr) P_n^m(\cos\theta) \cos m(\phi-\phi_o) , \tag{3a}$$

$$F_r = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \hat{C}_{m,n} r h_n^{(1)}(Kr) P_n^m(\cos\theta) \sin m(\phi-\phi_o) . \tag{3b}$$

The Spherical Bessel and Hankel functions $j_n(Kr)$ and $h_n^{(1)}(Kr)$, respectively are expressed in terms of the corresponding cylindrical functions in the following manner.[3]

$$j_n(Kr) = \sqrt{\pi/(2Kr)} \; J_{n+\frac{1}{2}}(Kr) \, , \tag{4a}$$

and

$$h_n^{(1)}(Kr) = \sqrt{\pi/(2Kr)} \; H^{(1)}_{n+\frac{1}{2}}(Kr) \tag{4b}$$

The tangential scattered electric fields are given by

$$E_\theta^s = -\frac{1}{r\sin\theta} \frac{\partial}{\partial\phi} F_r - \frac{1}{i\omega\epsilon_o} \frac{1}{r} \frac{\partial^2}{\partial r \partial\theta} A_r \, , \tag{5a}$$

$$E_\phi^s = \frac{1}{r}\frac{\partial}{\partial\theta} F_r - \frac{1}{i\omega\epsilon_o} \frac{1}{r\sin\theta} \frac{\partial^2}{\partial r \partial\phi} A_r \, . \tag{5b}$$

Now requiring that the total (incident and scattered) electric fields vanish at the surface r = a of the conducting sphere, the amplitude coefficients $\hat{B}_{m,n}$ and $\hat{C}_{m,n}$ can be determined in principle. As mentioned previously, only the final results, without the detailed derivation, are presented in the following:

$$\hat{B}_{m,n} = \frac{-iK\mathcal{J}(4\ell+2m+3)(2\ell+1)!\beta_{m+2\ell+1} H_m^{(1)}(K\rho_o)(2-\delta_{o,m})}{(2\ell+m+1)(2\ell+m+2)\cdot \ell!(\ell+m)!\, 2^{2\ell+m+2}} \times \delta_{n,m+2\ell+1} \, , \tag{6a}$$

where

$$\beta_n = -\left[Ka\, j_n(Ka)\right]' \Big/ \left[Ka\, h_n^{(1)}(Ka)\right]' \, . \tag{6b}$$

The prime sign indicates derivative with respect to Ka.

$$\hat{C}_{m,n} = \frac{-\mathcal{J}\omega\mu_o m(2\ell)!(4\ell+2m+1)\alpha_{m+2\ell} H_m^{(1)}(K\rho_o)}{(2\ell+m)(2\ell+m+1)\ell!(\ell+m)!\, 2^{2\ell+m+1}} \cdot \delta_{n,m+2\ell} \, , \tag{7a}$$

where

$$\alpha_n = -j_n(Ka) \Big/ h_n^{(1)}(Ka) \, . \tag{7b}$$

It should be noted here that although $E_\phi^{inc} \equiv 0$, $E_\phi^s \neq 0$. However, both $H_z^{inc}$ and $H_z^s$ vanish identically.

## REFERENCES


1. P.M. Morse, Vibration and Sound, Chapt. VII, McGraw Hill Book Co., New York, 1948.

2. P.M. Morse and H. Feshback, Methods of Theoretical Physics, Vol. II, Chapt. II, McGraw Hill Book Co., New York, 1953.