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RETRIEVAL OF THE ELECTRICAL RADIUS ka FROM THE SCATTERED FIELD IN CIRCULAR CYLINDRICAL AND SPHERICAL COORDINATES

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The inverse problem of electromagnetic scattering is considered for cylindrical and spherical scattering geometries. The incentive is to determine the shape of an unknown perfectly conducting scatterer from bistatic measured data for a given transmitted field. This, in general, requires the retrieval of the associated scattered near field from the measured far field. Combining the recovered near field with the unknown incident field, the unknown surface locus may then be specified by employing inverse scattering boundary conditions or methods of analytical continuation.[1]

As a first approach, though necessarily not the best choice, a series expansion of the scattered field in cylindrical(spherical)vector wave functions is employed where c and s denote cylindrical and spherical cases,respectively.[2,3] The associated expansion coefficients are determined by a matrix inversion technique which imposes severe restrictions on the distribution of the measurement aspect angles. The instability inherent in the matrix inversion procedure is caused by the particular properties of the employed vector wave expansion. These properties are studied in detail in the first part of this analysis. It is first shown that the transverse field components can be related to the unknown expansion coefficients in terms of the scattered field matrix. The radial dependence of the vector wave functions is extracted and combined with the associated expansion coefficients without employing the commonly applied far field approximation of the radial functions. The matrix elements thus represent the truncated set of vector surface harmonics. These elements are arranged so that a closed form solution of the determinants associated with the particular scattered field matrix considered can be obtained most efficiently,[2,3] where

$$|F(x=\cos\phi)|_c = |V(x,N)|, \quad v_{\mu\nu} = x_\mu^{\nu-1}$$

$$|F(\theta,\phi)|_s = \left|\begin{matrix}[G^1] & [G^2]\\ [G^2] & [G^1]\end{matrix}\right| \cdot \left(\prod_{\mu=1}^{N} \sin 2m\phi_\mu\right)$$

$$g^1_{\mu\nu} = \frac{m}{\sin\theta} P_\nu^m(\cos\theta_\mu), \quad m = \text{constant}$$

$$g^2_{\mu\nu} = \frac{\partial}{\partial\theta} P_\nu^m(\cos\theta_\mu) \quad , m = \text{constant}$$

Employing a novel series expansion of the associated Legendre's functions of the first kind, an analytical closed form solution of these determinants is derived using properties of the Vandermonde determinant. Neglecting constant multipliers, the closed form solution is given by

$$|F(\phi)|_c = \prod_{N\geq r>t\geq 1}(\cos\phi_r - \cos\phi_t)$$

$$|F(\theta,\phi|_s = \prod_{\mu=1}^{N} \sin 2m\phi_\mu \sin^{2m}\theta_\mu \prod_{N\geq r>t\geq 1}(\cos\theta_r - \cos\theta_t)^2$$

Since it is the aim to optimize the distribution of the measurement aspect angles chosen for computation, a novel optimization procedure for determinants of this particular type is derived[3] which is not found elsewhere in the literature. Employing fundamental theorems of Gauss and Vita on the factorized root expansion of polynomials of order n, the optimization function is presented in closed form solution.[2,3] The N distinct roots of this optimization function specify the optimum distribution of the truncated number of N measurement aspect angles

$$O_N(x_r)_c = (1 - x_r^2)^{\frac{1}{2}} P^1_{N-1}(x_r)$$

$$0_N^m(x_r)_s = (1 - x_r^2)^{\frac{1-m}{2}} P_{N+m-1}^{m-1}(x_r)$$

This important result is verified by computation and it is shown that the unknown expansion coefficients can be recovered for the mth degree multipole cases to the degree of accuracy dictated only by the order of truncation and by any suitable measurement technique. It is shown that optimal results are obtained if the measurement domain is centered about the 90° bistatic angle, which is consistent with the monobistatic equivalence theorem.[4] It is to be noted that a similar closed form solution of the determinant associated with non-symmetrical vector scattering geometries is derived and the cases N=3(m=0,1) and N=8(m=0,1 and 2) will be presented.

The ultimate aim of this analysis is to show that the electrical radius ka can be recovered from the given set of expansion coefficients without employing methods of analytical continuation or inverse scattering boundary conditions.[1] For the cases of perfectly conducting circular cylindrical or spherical scatterers, the hypothesis is set forth that all the information required to recover ka is explicitly contained in the set of electric (a_n) and magnetic (b_n) expansion coefficients,[2,3] where

$$(a_n)_c = -\frac{J_n(ka_c)}{H_n^{(1)}(ka_c)}, \quad (b_n)_c = -\frac{J_n'(ka_c)}{H_n^{(1)\prime}(ka_c)}$$

$$(a_n)_s = -\frac{j_n(ka_s)}{h_n^{(1)}(ka_s)}, \quad (b_n)_s = -\frac{[ka_s j_n(ka_s)]'}{[ka_s h_n^{(1)}(ka_s)]'}$$

Employing recurrence relations between three contiguous radial functions, it is shown that ka can be obtained from any four contiguous expansion coefficients in the electric(e) and the mixed electric-magnetic (e-h) polarization cases, where

$$(ka_c)_e^2 = 4(n-1)n\,\frac{(a_{n+1}-a_n)(a_{n-1}-a_{n-2})}{(a_{n+1}-a_{n-1})(a_n-a_{n-2})}.$$

$$(ka_c)_{e-h}^2 = (n-1)n\,\frac{(b_n-a_n)(b_{n-1}-a_{n-1})}{(b_n-a_{n-1})(b_{n-1}-a_n)}$$

$$(ka_s)_e^2 = (4n^2-1)\,\frac{(a_{n+1}-a_n)(a_{n-1}-a_{n-2})}{(a_{n+1}-a_{n-1})(a_n-a_{n-2})}$$

$$(ka_s)_{e-h}^2 = n^2\,\frac{(b_n-a_n)(b_{n-1}-a_{n-1})}{(b_n-a_{n-1})(b_{n-1}-a_n)}$$

Note that corresponding expressions are identical in form except for the constant multipliers. In the magnetic case (b_n), a more elaborate expression for the recovery of ka is obtained since no recursion relations between three contiguous derivatives of radial functions exist; hence recourse must be taken to the important relations between contiguous expansion coefficients of the electric and magnetic type as well as of degenerate relations of radial functions where

$$(b_n)_c = \frac{a_n(a_{n-1}+a_{n+1})-2a_{n-1}a_{n+1}}{2a_n-a_{n-1}-a_{n+1}}$$

$$(b_n)_s = \frac{n\,a_{n+1}(a_n-a_{n-1})-(n+1)a_{n-1}(a_{n+1}-a_n)}{n(a_n-a_{n-1})-(n+1)(a_{n+1}-a_n)}$$

$$a_1^c \equiv b_o,\ a_{o,2}^c \equiv f(b_o,b_1,\ldots,b_4.)$$

$$a_o^s \equiv -(1+b_o),\ a_1^s \equiv f(b_o,b_1,\ \ldots,b_3.)$$

Although the expressions derived are restricted to perfectly conducting circular cylindrical and spherical scatterers, the method is presently extended with success to recover the principal radii of curvature of other scatters of axial and rotational symmetry.

Finally, it is interesting to note that a unique relationship between the sets of electric and magnetic type expansion coefficients is derived which is valid in the Sommerfeld, Fresnel and Fraunhofer region of the scattered field [2,3]. It is anticipated that similar relations between the two associated types of vector functions may exist. If such relations could be proved, the inverse problem of scattering would be uniquely resolved in terms of only one set of vector wave functions.

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1 Weston, V.H. and Boerner, W.M., 1969, Can. Journ. Phys., 47, 1177.
2 Boerner, W.M., Vandenberghe, F.H. and Hamid, M.A.K., 1971, Can. Journ. Phys., 49, 804.
3 Boerner, W.M. and Vandenberghe, F.H. 1971, Can. Journ. Phys., 49, in press for May 15, 1971.
4 Kell, R.E., 1965, Proc.IEEE, 53,983.