DIFFRACTION OF A PLANE WAVE BY A SINUSOIDAL GRATING OF FINITE WIDTH — ASYMPTOTIC SOLUTIONS —

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1. Introduction

Analysis of the diffraction by gratings is important in electromagnetic theory and optics. Various analytical and numerical methods have been developed so far and the diffraction phenomena have been investigated for many kinds of gratings [1]-[5]. However, there are only a few papers that have treated the diffraction by gratings using function-theoretic methods. The Wiener-Hopf technique is known as a powerful tool for analyzing the diffraction problems, and the author has solved the diffraction by several transmission gratings of parallel-plate geometry using this technique [6][7]. He has also clarified that the gratings of this type exhibit useful features as resonators and filters from a technical point of view.

Most of analyses in the above-mentioned papers are restricted to the case of plane boundaries or periodic structures. It is important to investigate the diffraction by gratings without these restrictions. In [8], as an example of gratings with non-plane boundaries and finite periodicity, the author has treated the diffraction by a sinusoidal grating of finite width and developed a method of solution using the Wiener-Hopf technique combined with the perturbation method. Analysis of the problems of this kind is particularly important for investigating the edge effect of the grating with finite periodicity. In the present paper, we shall treat the same problem and derive the asymptotic solutions to the perturbed Wiener-Hopf equations obtained in the previous paper [8] and make a physical interpretation for the results briefly. The time factor is assumed to be $\exp(-i\omega t)$ and suppressed throughout this paper.

2. Exact solutions to the perturbed Wiener-Hopf equations

Let us consider the diffraction of an E-polarized plane wave by a sinusoidal grating of finite width as shown in Fig. 1. The grating surface is perfectly conducting and is described by the equation $x = h\sin mz$, $-a \leqq z \leqq a$, where h and m are some positive constants. Let the total electric field $\phi^t(x,z)$ $(\equiv E_y^t(x,z))$ be

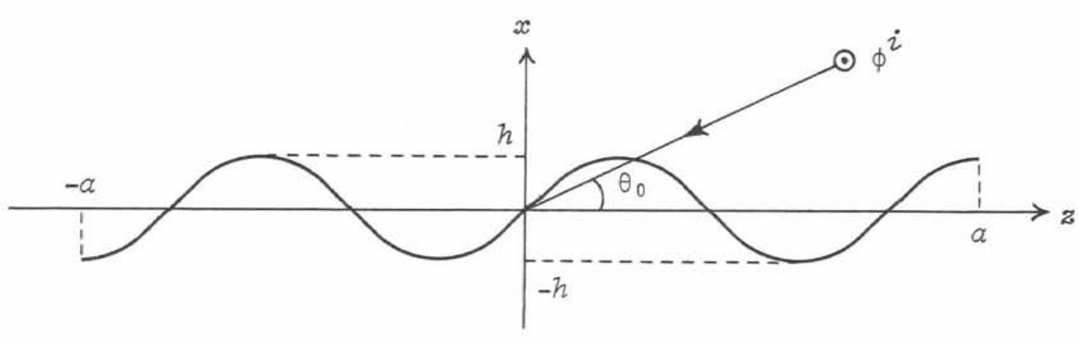

Fig. 1. Geometry of the problem.

$$\phi^t(x,z) = \phi^i(x,z) + \phi(x,z), \tag{1}$$

where $\phi^i(x,z)$ is an incident field given by

$$\phi^i(x,z) = e^{-ik(x\sin\theta_0 + z\cos\theta_0)}, \qquad 0 < |\theta_0| < \pi/2. \tag{2}$$

In (2), k $(=\omega\sqrt{\mu_0\varepsilon_0})$ is the free-space wave number. Assuming that the depth h of the grating is sufficiently small compared with the wavelength, we can approximate the boundary condition on the grating surface by

$$\phi^t(0,z) + h\sin mz\frac{\partial\phi^t(0,z)}{\partial x} + O(h^2) = 0, \qquad -a < z < a. \tag{3}$$

This shows that the original diffraction problem has now been reduced to that concerned with a flat strip under the mixed boundary condition (3).

For convenience of analysis, we assume that the medium is slightly lossy as in

$$k = k_1 + ik_2, \qquad 0 < k_2 \ll k_1. \tag{4}$$

Then the Wiener-Hopf procedure combined with the perturbation method [8] leads to the following Fourier integral representation for the scattered field :

$$\phi(x,z) = \phi^{(0)}(x,z) + h\phi^{(1)}(x,z) + O(h^2), \tag{5}$$

where

$$\left.\begin{aligned}\phi^{(0)}(x,z) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} C^{(0)}(\alpha)\, e^{\mp\gamma(\alpha)x - i\alpha z}\, d\alpha,\\ &\qquad\qquad\qquad\qquad x \gtrless 0,\\ \phi^{(1)}(x,z) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} C^{(1)}_{1,2}(\alpha)\, e^{\mp\gamma(\alpha)x - i\alpha z}\, d\alpha,\end{aligned}\right\} \tag{6}$$

$$C^{(0)}(\alpha) = e^{-i\alpha a}U_-(\alpha) + e^{i\alpha a}U_{(+)}(\alpha), \tag{7}$$

$$\begin{aligned}C^{(1)}_{1,2}(\alpha) = {} & e^{-i\alpha a}V_-(\alpha) + e^{i\alpha a}V_{(+)}(\alpha)\\ & \pm \frac{1}{4i}\Big[\frac{e^{-i(\alpha+m)a}U_-(\alpha+m) + e^{i(\alpha+m)a}U_{(+)}(\alpha+m)}{K(\alpha+m)}\\ & - \frac{e^{-i(\alpha-m)a}U_-(\alpha-m) + e^{i(\alpha-m)a}U_{(+)}(\alpha-m)}{K(\alpha-m)}\Big],\end{aligned} \tag{8}$$

$$K(\alpha) = \{2\gamma(\alpha)\}^{-1}, \qquad \gamma(\alpha) = (\alpha^2 - k^2)^{1/2}. \tag{9}$$

In (6)-(9), α $(\equiv \sigma + i\tau)$ is a complex variable and the proper branch for the double-valued function $\gamma(\alpha)$ is chosen such that $\gamma(\alpha)$ reduces to $-ik$ when $\alpha = 0$, and

$$\left.\begin{aligned}U_{\substack{(+)\\-}}(\alpha) &= (1/2)\{U^s_{(+)}(\pm\alpha) \pm U^d_{(+)}(\pm\alpha)\},\\ V_{\substack{(+)\\-}}(\alpha) &= (1/2)\{V^s_{(+)}(\pm\alpha) \pm V^d_{(+)}(\pm\alpha)\},\end{aligned}\right\} \tag{10}$$

where

$$U_{(+)}^{s,d}(\alpha) = \mp K_+(\alpha)\Big[\frac{A_0}{K_-(k\cos\theta_0)(\alpha+k\cos\theta_0)}$$

$$\mp \frac{B_0}{K_+(k\cos\theta_0)(\alpha-k\cos\theta_0)} - u_{s,d}(\alpha)\Big], \tag{11}$$

$$V_{(+)}^{s,d}(\alpha) = \mp K_+(\alpha)\Big[\frac{k\sin\theta_0}{2}\sum_{n=1}^{2}(-)^n\Big\{\frac{A_n}{K_-(k\cos\theta_n)(\alpha+k\cos\theta_n)}$$

$$\pm \frac{B_n}{K_+(k\cos\theta_n)(\alpha-k\cos\theta_n)}\Big\} - v_{s,d}(\alpha)\Big], \tag{12}$$

$$u_{s,d}(\alpha) = \frac{1}{\pi i}\int_k^{k+i\infty}\frac{e^{2i\beta a}\,U_{(+)}^{s,d}(\beta)}{K_-(\beta)(\beta+\alpha)}\,d\beta, \tag{13}$$

$$v_{s,d}(\alpha) = \frac{1}{\pi i}\int_k^{k+i\infty}\frac{e^{2i\beta a}\,V_{(+)}^{s,d}(\beta)}{K_-(\beta)(\beta+\alpha)}\,d\beta, \tag{14}$$

$$K_\pm(\alpha) = 2^{-1/2}\,e^{i\pi/4}\,(k\pm\alpha)^{-1/2}, \tag{15}$$

$$A_n = \frac{\exp(ika\cos\theta_n)}{(2\pi)^{1/2}\,i}, \quad B_n = \frac{\exp(-ika\cos\theta_n)}{(2\pi)^{1/2}\,i}, \quad n = 0, 1, 2, \tag{16}$$

$$\theta_1 = \cos^{-1}(\cos\theta_0 - m/k), \quad \theta_2 = \cos^{-1}(\cos\theta_0 + m/k). \tag{17}$$

Subscripts + and - introduced above imply that the functions are regular in the upper ($\tau > -k_2\cos\theta_0$) and lower ($\tau < k_2\cos\theta_0$) halves of the complex α-plane, respectively, whereas the subscript (+) is used for the functions regular in the upper half-plane except for simple poles at $\alpha = k\cos\theta_0$ or $\alpha = k\cos\theta_n$ for $n = 1, 2$. Equations (11) and (12) are the exact solutions to the perturbed Wiener-Hopf equations derived in [8], but they are formal in the sense that the branch-cut integrals with the unknown functions in their integrands are involved. It can be proved with the aid of the edge condition that $U_{(+)}^{s,d}(\alpha)$ and $V_{(+)}^{s,d}(\alpha)$ are $O(\alpha^{-3/2})$ as $\alpha \to \infty$ in the upper half-plane.

3. Asymptotic solutions

In this section, we shall derive the asymptotic solutions for large $|k|a$ based on the results obtained in the previous section. Let us introduce the auxiliary functions as in

$$F_+^{s,d}(\alpha) = U_{(+)}^{s,d}(\alpha) \pm \Big(\frac{A_0}{\alpha+k\cos\theta_0} \pm \frac{B_0}{\alpha-k\cos\theta_0}\Big). \tag{18}$$

Then $u_{s,d}(\alpha)$ defined by (13) can be expanded asymptotically for large $|k|a$ as

$$u_{s,d}(\alpha) \sim \frac{K_+(k)\{\chi_a(k) \pm \chi_b(k)\}}{1 \mp K_+(k)\,\xi(k)}\xi(\alpha) \pm A_0\,\eta_a(\alpha) + B_0\,\eta_b(\alpha) \tag{19}$$

after some manipulations, where

$$\xi(\alpha) = (e^{2ika}/\pi a^{1/2})\,\zeta_0\{-2i(\alpha+k)a\}, \tag{20}$$

$$\eta_{a,b}(\alpha) = \frac{e^{2ika}}{\pi a^{1/2}} \frac{\zeta_0\{-2i(\alpha + k)a\} - \zeta_0\{-2i(1 \pm \cos\theta_0)ka\}}{\alpha \mp k\cos\theta_0}, \tag{21}$$

$$\chi_a(\alpha) = A_0\,\eta_a(\alpha) + B_0\,P_b(\alpha), \qquad \chi_b(\alpha) = B_0\,\eta_b(\alpha) + A_0\,P_a(\alpha), \tag{22}$$

$$P_{a,b}(\alpha) = \frac{1}{\alpha \pm k\cos\theta_0}\left[\frac{1}{K_+(\alpha)} - \frac{1}{K_\mp(k\cos\theta_0)}\right], \tag{23}$$

$$\zeta_0(z) = \int_0^\infty \frac{t^{1/2} e^{-t}}{t+z}\,dt, \qquad |\arg z| < \pi. \tag{24}$$

Substituting (19) into (11) and taking (10) into account, the asymptotic expressions of $U_{(+)}(\alpha)$ and $U_-(\alpha)$ for large $|k|a$ are derived. Almost similar procedure can be applied to the asymptotic evaluation of $v_{s,d}(\alpha)$ defined by (14). Therefore, we can substitute this result into (12) to obtain the asymptotic representations of $V_{(+)}(\alpha)$ and $V_-(\alpha)$ for large $|k|a$ by using (10), but the detailed derivation is omitted here.

Collecting the results obtained above, it is found that $\phi^{(0)}(x, z)$ defined in (6) gives the field scattered by a flat strip without sinusoidal corrugation [9]. On the other hand, $\phi^{(1)}(x, z)$ is the correction term to $\phi^{(0)}(x, z)$ due to the presence of corrugation. Using a rigorous asymptotics, we can show that these terms are both of $O\{(ka)^{-3/2}\}$ as $|k|a \to \infty$.

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