Vertical Dipole Radiation in the Presence of the Spherical Earth: Discontinuity and Convergence Studies

Le-Wei Li[1], Ting Fei[1], Zhong-Cheng Li[1], and Qun Wu[2]

[1]*Department of Electrical and Computer Engineering*
National University of Singapore, Kent Ridge, Singapore 119260
[2]*Department of Electronics and Communications Engineering*
Harbin Institute of Technology, 92 Xidazhi Street, Harbin 150001, China

Abstract In this paper, we have re-looked into the classic problem of the vertical antenna radiation in the presence of a spherical earth which is considered to be either perfectly electric conducting or lossy dielectric. Three important issues were considered, one of which is the closed solution to the direct wave radiation into the observation point (that can be located anywhere in the space), the second of which is the continuity of the field expressions on the spherical surface at $r = r'$ in the space, and the third of which is the fast solution to the scattered field due to the presence of the earth. The scattered field is split into two contributions: one is the dominant and closed form solution while the other is the summated contribution. The second summated contribution is much smaller in value than the first dominant term and at the same time, it has better convergence property than the original scattered field in summation form. The convergence number is quantitatively analyzed and analytically obtained. Some examples are considered to demonstrate the special properties of the respective field contributions.

I Formulation of the Problem

A Basic Consideration

In this paper, we will consider a vertical dipole radiating in the presence of a spherical earth shown in Fig. 1. A time dependence $\exp(-i\omega t)$ is assumed for the fields throughout the paper. The dipole

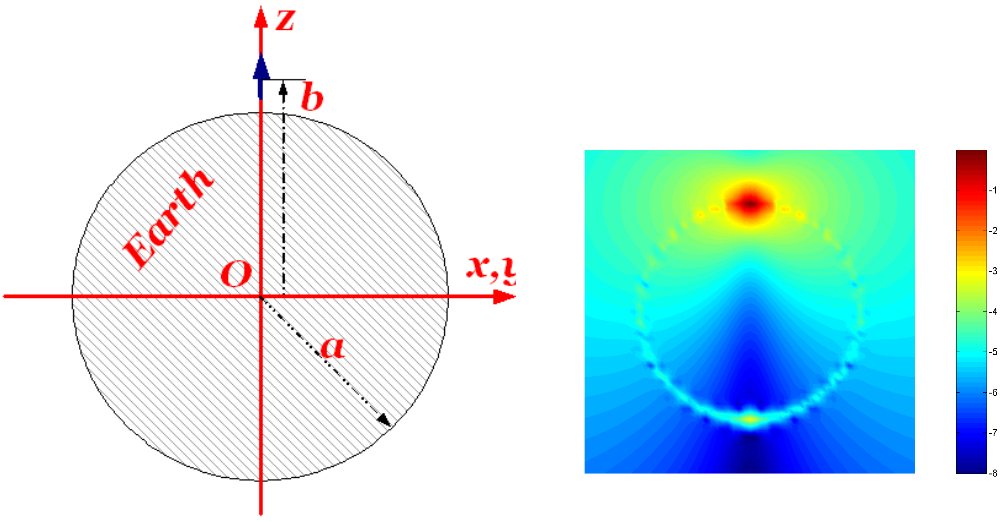

Figure 1: (a) Radiation of a vertical dipole in the presence of a spherical earth and (b) Radiation pattern of a vertical dipole in the free space obtained using the formula in discontinued field.

antenna takes the current distribution expressed under the spherical coordinates in the following form:

$$\boldsymbol{J}(\boldsymbol{r}') = I_0 \frac{\delta(r' - b)\delta(\theta')\delta(\phi')}{b^2 \sin\theta'} \widehat{\boldsymbol{r}}, \tag{1}$$

where I_0 denotes the amplitude of the current distribution while $\delta(\bullet)$ stands for the Dirac delta function, and b identifies the height of the dipole antenna from the spherical coordinate centre.

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II Field Expressions Associated with Spherical Earth

A Series Expressions

The field expressions due to the dipole antenna can be formulated for $r \gtrless b$ as follows:

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{k\omega\mu_0 c I_0}{4\pi(kb)} \sum_{n=1}^{\infty}(2n+1) \begin{cases} \Big\{j_n(kb) + B_N^{(11)} h_n^{(1)}(kb)\Big\}\boldsymbol{N}_{e0n}^{(1)}(k), \\ h_n^{(1)}(kb)\Big\{\boldsymbol{N}_{e0n}(k) + \mathcal{B}_N^{11} h_n^{(1)}(kb)\boldsymbol{N}_{e0n}^{(1)}(k)\Big\}, \end{cases} \tag{2}$$

where the coefficient $\mathcal{B}_N^{11}$ is given in [1].

It is seen that the radiated fields due to the vertical antenna consist of only the TM modes. It is also seen that the electromagnetic fields for the regions where $r > r'$ and $r < r'$ differ in expression. Numerically, it is verified that the discontinuity exists on the spherical surface where $r = r'$ regardless of the spherical angles of θ and ϕ, as seen in Fig. 1.

B Continuous Form of Field Expression

Physically, this is not correct as the discontinuity exists only in the source region. In the present case, the discontinuity, if any, should exists only at $\boldsymbol{r} = \boldsymbol{r}'$ instead of $r = r'$. Therefore, it is necessary to resolve this problem in the present work first.

As the discontinuity occurs in the contributions of the unbounded Green's function, it means that the discontinuity occurs in direct wave modes. To compute the field in Eq. (2) efficiently, we split the total field into the direct and reflected (or scattered) fields as follows:

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{\text{direct}}(\boldsymbol{r}) + \boldsymbol{E}_{\text{scat}}(\boldsymbol{r}) \tag{3}$$

where

$$\boldsymbol{E}_{\text{direct}}(\boldsymbol{r}) = -\frac{k\omega\mu_0 I_0}{4\pi(kb)} \sum_{n=0}^{\infty}(2n+1) \begin{cases} j_n(kb)\boldsymbol{N}_{e0n}^{(1)}(k), & r > b; \\ h_n^{(1)}(kb)\boldsymbol{N}_{e0n}(k), & r < b; \end{cases} \tag{4a}$$

$$\boldsymbol{E}_{\text{scat}}(\boldsymbol{r}) = -\frac{k\omega\mu_0 I_0}{4\pi(kb)} \sum_{n=0}^{\infty}(2n+1)\mathcal{B}_N^{11} h_n^{(1)}(kb)\boldsymbol{N}_{e0n}^{(1)}(k). \tag{4b}$$

To resolve the discontinuity problem, we will employ the following series expansion for $r \gtrless b$:

$$\frac{e^{ikR}}{ikR} = \sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta) \begin{cases} j_n(kb)h_n^{(1)}(kr), \\ h_n^{(1)}(kb)j_n(kr); \end{cases} \tag{5}$$

where the distance R is defined as $R = |\boldsymbol{r} - \boldsymbol{r}'| = \sqrt{b^2 + r^2 - 2br\cos\theta}$. Now, we consider the following operator $\frac{1}{k}\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times [\bullet \boldsymbol{r}]$ and take its operation on both sides of Eq. (5), we have

$$\boldsymbol{E}_{\text{direct}}(\boldsymbol{r}) = -\frac{k\omega\mu_0 I_0}{4\pi(kb)} \frac{1}{k} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \left[\frac{e^{ikR}}{ikR}\boldsymbol{r}\right]. \tag{6}$$

Consider the following scalar Green's function

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{e^{ikR}}{ikR} = \frac{-ie^{ik\sqrt{b^2+r^2-2br\cos\theta}}}{k\sqrt{b^2+r^2-2br\cos\theta}}. \tag{7}$$

We have carefully checked the convergence number and found that the convergence number is determined by $N_{\text{iter}} = x + 3\sqrt{x} + 2$ where $x = ka$ whose a is the radius of the sphere.

Taking the curl of the Green's function with a radial vector $\boldsymbol{r}$, we have

$$\boldsymbol{\nabla} \times [G(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{r}] = -\frac{br\sin\theta e^{ik\sqrt{b^2+r^2-2br\cos\theta}}}{k[b^2+r^2-2br\cos\theta]^{\frac{3}{2}}} \left[i + k\sqrt{b^2+r^2-2br\cos\theta}\right]. \tag{8}$$

Further taking the curl of both sides of Eq. (8) and dividing it by the factor of $1/k$, we can derive the field normalized by the coefficient in the front of the summation sign in Eq. (4a), i.e., $-(k\omega\mu_0 cI_0)$ $/(4\pi kb)$. In scalar form, we have the following explicit expressions of electric field components:

$$\begin{aligned}
E_{n,r} &= \frac{1}{k}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times[G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{r}]_{r-\text{component}} \\
&= \frac{be^{ik\sqrt{b^2+r^2-2br\cos\theta}}}{k^2[b^2+r^2-2br\cos\theta]^{\frac{5}{2}}}\Bigg\{4br\cos^2\theta\left[i+k\sqrt{b^2+r^2-2br\cos\theta}\right]+br\Big[3i+k\Big(-ik\left(b^2+r^2\right) \\
&\quad +3\sqrt{b^2+r^2-2br\cos\theta}\Big)\Big]\sin^2\theta+(2i)\cos\theta\Big[i\left(b^2+r^2\right)\left(i+k\sqrt{b^2+r^2-2br\cos\theta}\right) \\
&\quad +b^2k^2r^2\sin^2\theta\Big]\Bigg\}, \qquad (9a)
\end{aligned}$$

$$\begin{aligned}
E_{n,\theta} &= \frac{1}{k}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times[G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{r}]_{\theta-\text{component}} \\
&= \frac{be^{ik\sqrt{b^2+r^2-2br\cos\theta}}}{k^2[b^2+r^2-2br\cos\theta]^{\frac{5}{2}}}\Bigg\{i\Big[-r^2+k^2r^4+2b^2\left(1+k^2r^2\right)\Big]-ibr\cos\theta\Big[1+k\Big(b^2k+3kr^2 \\
&\quad -i\sqrt{b^2+r^2-2br\cos\theta}\Big)\Big]+k\Big[\left(2b^2-r^2\right)\sqrt{b^2+r^2-2br\cos\theta}+ib^2kr^2\cos(2\theta)\Big]\Bigg\}\sin\theta. \qquad (9b)
\end{aligned}$$

The third component $E_{n,\phi}$ is found to vanish. The final expression of the direct-wave electric field is expressed in component form as follows:

$$E_{\text{direct},r}(\boldsymbol{r}) = -\frac{k\omega\mu_0 I_0}{4\pi(kb)}E_{n,r}, \qquad E_{\text{direct},\theta}(\boldsymbol{r}) = -\frac{k\omega\mu_0 I_0}{4\pi(kb)}E_{n,\theta}. \qquad (10)$$

At this moment, we have already obtained the direct wave in closed form in Eq. (4a). This form does not have the previous problem of the discontinuity at $r=r'$ but $\theta\neq\theta'$ or $\phi\neq\phi'$. Also, the procedure of deriving the expression is pretty simple and straightforward, which depicts apparently as an advantage of the present method over some of the existing ones.

III Convergence Issues

From the analysis of the terms in the normalized electric field expression, it is apparent that to directly sum up these terms is not an efficient approach. Therefore, it is necessary to have an alternative approach. To do so, we propose to extract the contribution of the electric field due to an image dipole of the real dipole in the presence of a large-radius sphere. The image location of a real source at $r=b$ can be approximated as at $r=2a-b$. Employing the scalar Green's function expression, we will have, for $r \gtrless 2a-b$, the following expression where $R'=\sqrt{r^2-2r(2a-b)\cos\theta+(2a-b)^2}$

$$\frac{1}{k}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\left[\frac{e^{ikR'}}{ikR'}\boldsymbol{r}\right] = \sum_{n=0}^{\infty}(2n+1)\begin{cases} j_n[k(2a-b)]\boldsymbol{N}^{(1)}_{e0n}(k), \\ h^{(1)}_n[k(2a-b)]\boldsymbol{N}_{e0n}(k). \end{cases} \qquad (11)$$

In practice, the upper level condition is almost always satisfied when the observation point is located in the air region (or outside of the sphere). Therefore, we will take the upper line for our subsequent formulation. Therefore, we have $\boldsymbol{E}_{\text{scat}}(\boldsymbol{r}) = \boldsymbol{E}_{\text{image}}(\boldsymbol{r}) + \boldsymbol{E}_{\text{corr}}(\boldsymbol{r})$ where the contributions due to its approximate image and the correction term are given respectively as

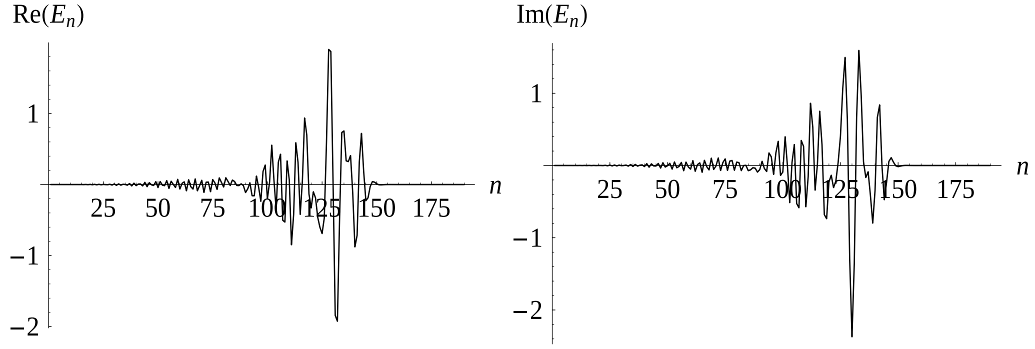
$$\boldsymbol{E}_{\text{image}}(\boldsymbol{r}) = -\frac{k\omega\mu_0 I_0}{4\pi(kb)}\frac{1}{k}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\left[\frac{e^{ikR'}}{ikR'}\boldsymbol{r}\right] - \frac{k\omega\mu_0 I_0}{4\pi(kb)}\boldsymbol{E}_n\Big|_{b\to(2a-b)}, \qquad (12a)$$

$$\boldsymbol{E}_{\text{corr}}(\boldsymbol{r}) = -\frac{k\omega\mu_0 I_0}{4\pi(kb)}\sum_{n=0}^{\infty}(2n+1)\Big[\mathcal{B}^{11}_N h^{(1)}_n(kb) - j_n[k(2a-b)]\Big]\boldsymbol{N}^{(1)}_{e0n}(k). \qquad (12b)$$

It is seen clearly that the scattered wave can be expressed as sum of two contributions: one is $\boldsymbol{E}_{\text{image}}(\boldsymbol{r})$ contributed by the approximate image source or dipole as in Eq. (12a) while the other is by the correction term $\boldsymbol{E}_{\text{corr}}(\boldsymbol{r})$ which is employed to achieve high accuracy of the solution as in Eq. (12b). It should be pointed out that

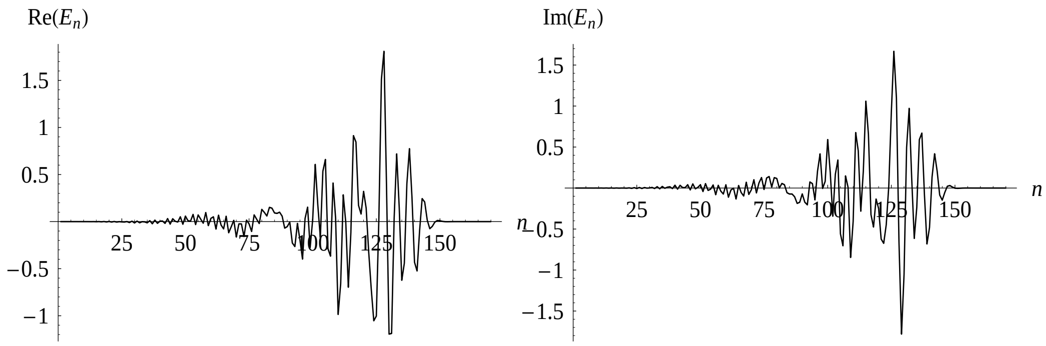
(a) The magnitude of $\boldsymbol{E}_{\text{image}}(\boldsymbol{r})$ is much larger than that of $\boldsymbol{E}_{\text{corr}}(\boldsymbol{r})$ so the dominant term has been extracted out and the $\boldsymbol{E}_{\text{corr}}(\boldsymbol{r})$ in Eq. (12b) is included as a correction to enhance the computational accuracy; and

(b) The convergence of $\boldsymbol{E}_{\text{corr}}(\boldsymbol{r})$ in Eq. (12b) is improved as compared with that originally given in Eq. (4b). Fig. 2 and Fig. 3 shows clearly that the real and imaginary parts of the scattered wave, after the dipole image contribution is extracted, converge in a narrower range of n.



(a) Real value of $E_n(n)$ (b) Imaginary value of $E_n(n)$

Figure 2: Convergence pattern of the normalized scattered electric field $E_n(n)$ after extraction of the dipole image contribution as a function of n for a perfectly conducting sphere.



(a) Real value of $E_n(n)$ (b) Imaginary value of $E_n(n)$

Figure 3: Convergence pattern of the normalized scattered electric field $E_n(n)$ after extraction of the dipole image contribution as a function of n for a dielectric lossy sphere.

IV Conclusion

Three important issues were considered in this paper: (1) the closed solution to the direct wave radiation into the observation point (that can be located anywhere in the space), (2) the continuity of the field expressions on the spherical surface at $r = r'$ in the space, and (3) the fast solution to the scattered field due to the presence of the earth. The scattered field is split into two contributions, that is, the dominant and closed form solution and the summated contribution. The second summated contribution is much smaller in value than the first dominant term and at the same time, it has better convergence property than the original scattered field in summation form. The convergence number is quantitatively analyzed and analytically obtained.

[1] Le-Wei Li, Pang-Shyan Kooi, Mook-Seng Leong, and Tat-Soon Yeo. Electromagnetic dyadic Green's function in spherically multilayered media. *IEEE Trans. Microwave Theory Tech.*, 42(12):2302–2310, December 1994. Part A.