ARRAY SYNTHESIS BY USE OF THE THEORY OF APPROXIMATION IN THE COMPLEX DOMAIN

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1. Introduction

It is known that the theory of approximation in the real domain was applied to the synthesis of linear array [1] [2]. In this paper, a new method of synthesis is firstly proposed which is based on the theory of approximation in the complex domain. First, the synthesis of linear array is considered. It can be seen that the method is simpler than the inverse Z-transform method. Second, this method is developed for the planar array arranged in rectangular grid.

2. Linear Array

Consider a typical array located in free space. Assume that the element patterns are the same, and I_n and α_n are the amplitude and phase of the nth element, respectively, then the far field is given by

$$E(\theta,\Phi)=f(\theta,\Phi)\sum_{n=0}^{N-1} I_n \exp\left[j\ knd\ \cos\theta+j\alpha_n\right] \tag{1}$$

where $f(\theta,\Phi)$ is the element pattern, $k=2\pi/\lambda$, λ is the free-space wavelength and d is the distance between two adjacent elements. It is known that

$$F(\theta,\Phi)=\sum_{n=0}^{N-1} a_n \exp\left[j\ knd\ \cos\theta\right] \tag{2}$$

is called the array factor or array pattern, where

$$a_n=I_n \exp\left[j\ \alpha_n\right] \tag{3}$$

Let

$$u=kd\ \cos\theta, \qquad z=\exp[ju] \tag{4}$$

Then, Eq(2) can be rewritten as

$$F(z)=\sum_{n=0}^{N-1} a_n z^n \tag{5}$$

It can be seen that F(z) is an analytic function.

For the given pattern F(z), it can be assumed that

$$\int_{|z|=1} |F(z)|^2 |dz| < \infty \tag{6}$$

Now, according to the theory of approximation in the complex domain, we know that F(z) can be expressed as follows:

$$F(z)=\sum_{n=0}^{\infty} C_n \Phi_n(z) \tag{7}$$

If the Szegò' s polynomials are selected as the basis functions, then it can be proved that [3]

$$F(z)=\sum_{n=0}^{\infty} a_n z^n \tag{8}$$

where

$$a_n=F^{(n)}(0)/n\,! \tag{9}$$

It implies that the following formula will provide the best mean-square approximation in the complex domain,

$$F(z)=\sum_{n=0}^{N-1} a_n z^n \tag{10}$$

where a_n is given by Eq. (9). The error is detemined by

$$\mu_N=\int_{|Z|=1}|F(z)|^2|dz|-2\pi\sum_{n=0}^{N-1}|a_n|^2 \tag{11}$$

If the limit of error is given by ε, then

$$\int_{|Z|=1}|F(z)|^2|dz|-2\pi\sum_{n=0}^{N-1}|a_n|^2<\varepsilon \tag{12}$$

Therefore, the total number of elements N can be determined by Eq. (12).

Now, if F(z) is the desired pattern, then the procedure of synthesis will be stated as follows:

(a) finding F(z) by using the following transform

$$u=-j\ln z \tag{13}$$

(b) calculating the coefficients a_n

(c) to determine the total number of elements N.

If the power pattern $|F(u)|^2$ is given, then the function $|F(z)|^2$ should be seperated into two parts, F(z) and $\overline{F(z)}$.

3. Planar Array (Rectangular Grid Arrangement)

Assume that the pattern of planar array is given by

$$F(\theta,\Phi)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} I_{mn}\{\exp[j(mk\,d_x\sin\theta\cos\varphi)+j\alpha_m]$$
$$\exp[j(n\,k\,d_y\sin\theta\sin\varphi)+j\,\beta_n]\} \tag{14}$$

where I_{mn} and $\alpha_m+\beta_n$ are the amplitude and phase of the element, respectively, with the coordinate (md_x, nd_y). It is seen that Eq. (14) can be rewritten as

$$F(u,v)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} a_{mn}\exp[jmu]\exp[jnv] \tag{15}$$

where

$$a_{mn} = I_{mn} \exp\left[j(\alpha_m + \beta_n)\right] \tag{16}$$

$$u = kd_x \sin\theta \cos\varphi \tag{17}$$

$$V = kd_y \sin\theta \sin\varphi \tag{18}$$

Let

$$z_1 = \exp\left[ju\right], \qquad z_2 = \exp\left[jv\right] \tag{19}$$

or

$$u = -j \ln z_1, \qquad v = -j \ln z_2 \tag{20}$$

Then, Eq. (15) becomes

$$F(z_1, z_2) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} a_{mn} z_1^m z_2^n \tag{21}$$

Obviously, $F(z_1, z_2)$ is an analytic function.

For the given pattern $F(z_1, z_2)$, it can be assumed that

$$\int_{|z_1|=1} \int_{|z_2|=1} |F(z_1, z_2)|^2 |dz_1||dz_2| < \infty \tag{22}$$

Thus, according to the theory of functions of several complex variables, we can prove that

$$F(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} z_1^m z_2^n \tag{23}$$

where

$$a_{mn} = \frac{1}{m!\, n!} \left[\frac{\partial^{m+n} F(z_1, z_2)}{\partial z_1^m \partial z_2^n} \right]_{\substack{z_1=0 \\ z_2=0}} \tag{24}$$

The following formula will provide the best mean-square approximation in the complex domain,

$$F(z_1, z_2) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} a_{mn} z_1^m z_2^n \tag{25}$$

where a_{mn} is given by Eq. (24). It is seen that this synthesis method of planar array is similar to the above synthesis method of linear array.

4. Examples

Example 1. Assume that the power pattern of a linear array is given by

$$|F(u)|^2 = \exp[-4(1-\cos u)^2]$$

Now, by using the transform $u = -j \ln z$, we have

$$|F(z)|^2 = \exp\left[-(z^2 - 4z + 3)\right] \overline{\exp\left[-(z^2 - 4z + 3)\right]}$$

So, we obtain

$$F(z) = \exp[-(z^2 - 4z + 3)]$$

If $N = 5$, then

$$a_0 = e^{-3}, \quad a_1 = 4e^{-3}, \quad a_2 = 7e^{-3}$$

$$a_3 = (20/3)\, e^{-3}, \quad a_4 = (19/6)e^{-3}$$

Example 2. Assume that

$$|F(u,v)| = |\sin\frac{Nu}{2}/\sin\frac{u}{2}|\,|\sin\frac{Nv}{2}/\sin\frac{v}{2}|$$

So, we have

$$F(u,v) = A[\sin\frac{Nu}{2}/\sin\frac{u}{2}][\sin\frac{Nv}{2}/\sin\frac{v}{2}]$$

where $|A| = 1$. According to Eq. (20), we obtain

$$F(z_1,z_2) = \frac{A}{z_1^{\frac{N-1}{2}} z_2^{\frac{N-1}{2}}} \frac{z_1^N - 1}{z_1 - 1} \frac{z_2^N - 1}{z_2 - 1}$$

Let $A = z_1^{\frac{N-1}{2}} z_2^{\frac{N-1}{2}}$. Obviously, $|A| = 1$ for $|z_1| = 1$ and $|z_2| = 1$. Now, we have

$$F(z_1,z_2) = \frac{z_1^N - 1}{z_1 - 1} \frac{z_2^N - 1}{z_2 - 1} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} z_1^m z_2^n$$

It can be seen that

$$a_{mn} = I_{mn} \exp[j(\alpha_m + \beta_n)] = 1$$

5. Conclusions

In this paper a simpler method to synthesise the planar array arranged in rectangular grid has been shown. From Eq.(12), it can be seen that the approximation error can easily be estimated by this method. Therefore when the error limit is given the number of the array can be determined more easily than the inverse Z-transform method [5].

References

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