

INVERSE VELOCITY ANALYSIS OF A GAUSSIAN PULSE PROPAGATION IN A
LORENTZ MEDIUM

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1 Introduction

The analysis of wave propagation in dispersive media was initiated by Sommerfeld and Brillouin[1] as a verification of Einstein's special relativity. They proved that no wave can propagate faster than light, but gave little about the physical meaning of superluminal group velocity. Consequently, the superluminal group velocity had been taken by almost all physicists as physically meaningless concept [2].

Against this, Garrett and McCumber[3] proved that a Gaussian pulse can travel at a velocity equal to the group velocity even when it is superluminal or negative. Although their result is remarkable, their analysis can be applied only to the initial stage of the propagation. Tanaka et al.[4] analyzed the propagation of Gaussian wave packet in a Lorentz medium to investigate more deeply the physical meaning of group velocity in the anomalous dispersion band. However, the use of the saddle point method seems to have prevented them from carrying out the precise analysis of the important early stage propagation.

Another problem of the past works for the investigation of the propagation velocity is to have given much more weight to the group velocity than to the energy velocity. In order to recognize the subtle difference between the group and energy velocities, it is very useful to use inverse velocity (will be called index) instead of velocity itself.

In this paper we shall show that the combination of high precision numerical Laplace transformation[5] and the use of inverse velocity give deeper insight into the propagation velocity.

2 Propagation of a Gaussian wave packet in a Lorentz medium

we shall consider the propagation of Gaussian wave packet in a single resonant Lorentz medium whose electric susceptibility is given by

$$\chi(s) = \omega_p^2 / (s^2 + 2\gamma s + \omega_0^2) \quad (1)$$

where ω_p is the plasma frequency, γ the loss factor, and ω_0 the resonant frequency of the Lorentz medium.

Using $\chi(s)$, the relative permittivity $\epsilon(s)$, the refractive index $n(s)$, the propagation function $K(s)$, the attenuation constant α , the group index I_g , and the energy index⁽¹⁾ I_e are expressed as

$$\begin{aligned} \epsilon(s) &= 1 + \chi(s), & n(s) &= \sqrt{\epsilon(s)} \\ K(s) &= sn(s)/c, & \alpha(\omega) &= [\Re[K(s)]]_{s=i\omega+0} \end{aligned}$$

⁽¹⁾Our I_e is expressed by $\chi(s)$, and a little different from that of Ref[1].

$$\begin{aligned}
I_g(\omega) &:= c/v_g = [\Re[\partial_s K(s)]]_{s=i\omega+0} \\
I_e(\omega) &:= c/v_e = [(1 + |\epsilon(s)| + |\partial_s[s\chi(s)]|)/\Re[(2n(s))]_{s=i\omega+0}
\end{aligned} \tag{2}$$

where c is the velocity of light, v_g the group velocity, v_e the energy velocity, ∂_s the differentiation with respect to s , and $s = i\omega + 0$ means that the frequency ω on the imaginary axis is approached from the right-hand side.

If $[t, x]$ denotes the space-time coordinates, the propagation of the wave packet is described by the following s-domain equation:

$$F(s, x) = F(s, 0)e^{-xK(s)} \tag{3}$$

where $F(s, 0)$ is the Laplace transform of the pulse at the starting point $[t, x] = 0$.

Genuine Gaussian pulse (gGp) has no initial and end points, so that, when we express it numerically, we must use some approximation which has an initial and an end point. We shall use here the m -th pseudo Gaussian pulse defined by

$$pgp(t, m, \sigma) := \sin(t/(\Delta\sqrt{m}))^n \tag{4}$$

where m is a positive integer called the order of gpg, and Δ is the standard deviation of the corresponding gGp. The 36th pgp is sufficient for our purpose, so we shall use the initial pulse

$$f(t, 0) = pgp(t, 36, \Delta) \exp(\omega_c t) \tag{5}$$

where ω_c is the carrier frequency.

3 Results and discussion

For simplicity, we shall use the following normalized variables, taking the resonant frequency as basis:

$$\Omega_p := \omega_p/\omega_0, \quad G := \gamma/\omega_0, \quad \Omega_c := \omega_c/\omega_0, \quad D := \Delta\omega_0, \quad X := x\omega_0/c, \quad T := \omega_0 t, \quad \Omega := \omega/\omega_0$$

Because the results depend critically on the values of parameters, for clear discussion it is necessary to fix the parameters. According to Ref.[4], we take the following parameters:

$$\Omega_p = 0.1, \quad G = 0.02, \quad \Omega_c = 1, \quad D = 100$$

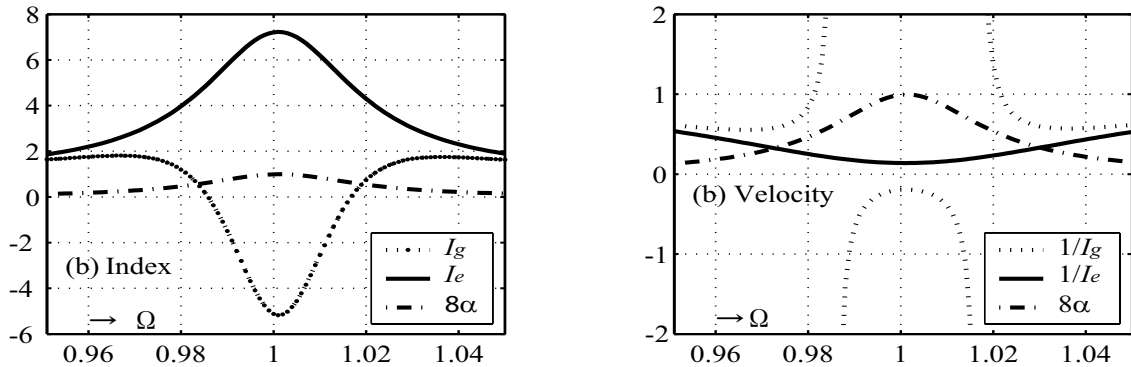


Figure 1: (a) Index representation, (b) Velocity representation

In Fig.1(a) we give the group index I_g , the energy index I_e , and attenuation constant (8 times enlarged) of our Lorentz medium. The following two items are of special importance for the later discussion.

- (1) At each frequency the energy index is always greater than the group index.
- (2) The group index has a maximum (≈ 1.8)

These important informations are difficult to obtain from the figure represented by velocities (see Fig.1(b)).

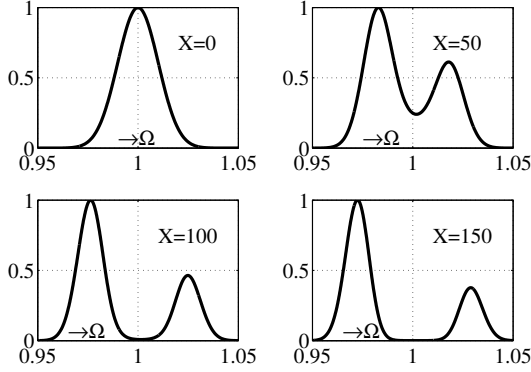


Figure 2: The spectra of signal at $X = 0, 50, 100, 150$

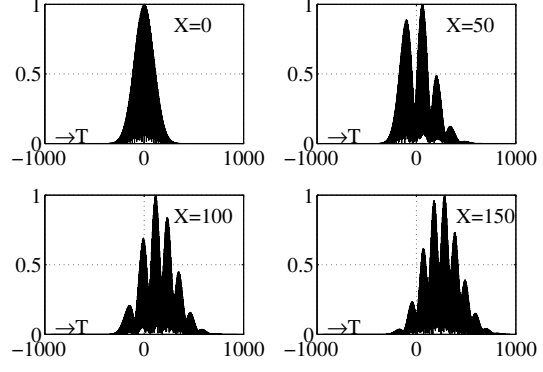


Figure 3: The waveforms of the signal at $X = 0, 50, 100, 150$

Fig.2 shows the spectra at $X = 0, 50, 100, 150$. Because of the large attenuation of the carrier frequency, the spectra become double-humped, resulting in complex waveforms (see Fig.3). The peaks of the pulse move at speeds different from that of the body of pulse.

Accordingly, the velocity of the peak point cannot be used to define the velocity of signal. To overcome this situation, we define the arrival time T_b and the index (inverse velocity)) of the barycenter of the pulse by

$$\begin{aligned}
 T_b(X) &:= \int T |f(T, X)|^2 dT / \int |f(T, X)|^2 dT \\
 I_b &:= \partial_X T_b(X)
 \end{aligned} \tag{6}$$

Fig.4 (a) shows the trajectory of barycenter of the pulse where, contrary to convention, the arrival time T_b of pulse is plotted as a function of the distance X . Fig. (b) is the corresponding barycenter index. We can see that the barycenter index becomes greater than the maximum (≈ 1.8) of the group index after $X \approx 35$. This is very important because it implies that we can no longer explain the propagation by group velocity only.

In order to get a better understanding, we shall use the mean group and energy indexes and the mean attenuation (denoted by J_g , J_e , and J_a respectively) defined by

$$\begin{aligned}
 J_g(X) &:= \int I_g(\Omega) |F(i\Omega, X)|^2 d\Omega / \int |F(i\Omega, X)|^2 d\Omega \\
 J_e(X) &:= \int I_e(\Omega) |F(i\Omega, X)|^2 d\Omega / \int |F(i\Omega, X)|^2 d\Omega \\
 J_a(X) &:= \int \alpha(\Omega) |F(i\Omega, X)|^2 d\Omega / \int |F(i\Omega, X)|^2 d\Omega
 \end{aligned} \tag{7}$$

We have plotted J_g , J_e , and $J_e - 24J_a$ in Fig.4 (b). We can see: (1) the initial barycenter index I_b is equal to the mean group index $J_g \approx -3.6$, (2) soon after, it increases more rapidly than J_g , (3) it exceeds the maximum of the group index (≈ 1.8) and approaches the mean

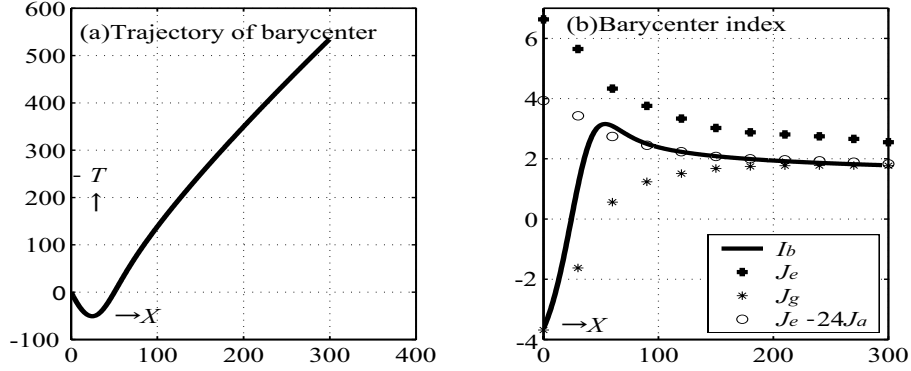


Figure 4: (a) The trajectory of barycenter of pules, (b) the barycenter index

energy index J_e , (4) after that, it decreases gently in parallel with J_e ; it agrees very well with the curve $J_e - 24J_a$.

These results show that there are cases which cannot be explained by the group velocity and that the energy velocity and attenuation are more essential.

According to our interpretation, every frequency component of the pulse moves at its own energy velocity which was given by Brillouin and never exceeds the light velocity even when the group velocity is superluminal.

However, roughly speaking, a pulse can make its barycenter move faster than the energy velocity by discarding or attenuating some of its rear part. If the attenuation is large enough, the barycenter velocity can become superluminal or negative. This is why a superluminal propagation always accompany a large attenuation.

The large attenuation of early stage in Fig.4(b) makes the initial barycenter velocity negative or superluminal. However, because the frequency components with larger attenuation decrease more rapidly leaving the frequency components with smaller attenuation, the mean attenuation becomes smaller.

Accordingly, the rate of discarding the rear part decreases and the barycenter velocity approaches the energy velocity. At this stage, it is clear that the barycenter velocity is mainly controlled by the energy velocity and the attenuation and cannot be explained by the group velocity.

As the attenuation becomes still smaller the discarding of the rear part becomes smaller, which means that the energy, group, and barycenter velocities become all equal.

References

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