DECOMPOSITION OF MOMENT METHOD IMPEDANCE MATRIX ELEMENTS FOR PRINTED WIRE ANTENNAS

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1. Introduction

This paper is a sequel to [1], which describes the moment method impedance matrix element $Z_{m,n}$ (composed of $g_{m-1, n-1}$, $g_{m-1, n}$, $g_{m, n-1}$, and $g_{m, n}$, all involving triple integrals) used for obtaining the current distribution along an arbitrarily shaped wire antenna printed on a dielectric substrate. To reduce the computational difficulty of $Z_{m, n}$, this paper presents a new impedance matrix element $Z^{NEW}_{m,n}$, consisting of single-, double-, and triple-integral terms.

2. Basic integral equation for a printed wire

Consider an arbitrarily shaped wire printed on a dielectric substrate of relative permittivity ε_r and thickness B. The wire is assumed to be thin and perfectly conducting. The current I (s') along the wire is obtained by solving the integral equation [1]:

$$\int_{\text{wire}} I(s') \left\{ -\frac{\partial}{\partial s} \frac{\partial}{\partial s'} (\Pi^{s}(s,s') - \Pi(s,s')) + k_{0}^{2}(\hat{s} \cdot \hat{s}') \Pi^{s}(s,s') \right\} ds' = -E_{\tan}^{i}(s)$$
(1)

where s (= s (x, y, z)) and s' (= s'(x', y', z' = B)) are the distances measured from the wire origin to an observation point with coordinates (x, y, z) and a source point with coordinates (x', y', z' = B) along the wire, respectively; \hat{s} and \hat{s} ' are the tangential unit vectors at the observation and source points, respectively; $k_0^2 = \omega^2 \mu_0 \epsilon_0$; E_{tan}^i (s) is the tangential component of an electric field impressed at the observation point; and Π^s (s (x, y, z), s'(x', y', B)) and Π (s (x, y, z), s'(x', y', B)) are Green's functions defined as

$$\Pi^{s}(S(x, y, z), S'(x', y', B)) = 2q \lim_{z \to B} \int_{0}^{\infty} J_{0}(\rho\lambda) e^{-u_{0}(z-B)} \frac{\lambda}{D_{e}(\lambda)} d\lambda$$
(2)

$$\Pi \left(S \left(x, y, z \right), S'(x', y', B) \right) = 2q \left(\epsilon_r - 1 \right) \lim_{z \to B} \int_{0}^{\infty} J_0(\rho \lambda) e^{-u_0 \left(z - B \right)} \frac{u_0 \lambda}{D_m(\lambda) D_e(\lambda)} d\lambda$$
(3)

where $q = -j/(4\pi\omega\epsilon_0)$; $J_0(\rho\lambda)$ is a Bessel function of the first kind of order zero with $\rho^2 = (x - x')^2 + (y - y')^2$; $D_e(\lambda) = u_0 + u_e \operatorname{coth} u_e B$ and $D_m(\lambda) = u_0\epsilon_r + u_e \tanh u_e B$ with $u_0 = (\lambda^2 - k_0^2)^{1/2}$ and $u_e = (\lambda^2 - \epsilon_r k_0^2)^{1/2}$.

 Π s and Π are expressed as

$$\Pi^{s} = \Psi^{s} + \Delta \Psi^{s}$$
(4)
$$\Pi = \Psi + \Delta \Psi$$
(5)

and

where $\psi^s = q \lim_{z \to B} e^{-jk_0R} / R$ and $\psi = \tau \psi^s$ with $R^2 = \rho^2 + (z-B)^2$ and $\tau = (\epsilon_r - 1)/(\epsilon_r + 1)$, and

$$\Delta \psi^{s} = q \lim_{z \to B} \int_{0}^{\infty} J_{0}(\rho \lambda) e^{-u_{0}(z-B)} \frac{\lambda}{u_{0}} [\frac{2u_{0}}{D_{e}(\lambda)} - 1] d\lambda$$
(6)

$$\Delta \Psi = q \lim_{z \to B} \int_{0}^{\infty} J_{0}(\rho \lambda) e^{-u_{0}(z-B)} \frac{\lambda}{u_{0}} \left[\frac{2(\varepsilon_{r}-1)u_{0}^{2}}{D_{m}(\lambda)D_{e}(\lambda)} - \tau \right] d\lambda$$
(7)

Using Eqs. (4) and (5), the left side of Eq. (1) is decomposed into four terms: $E_{\psi s} + \Delta E_{\psi s} + E_{\psi} + \Delta E_{\psi}$ = $-E_{tan}^{i}(s)$, where

$$E_{\psi s} = \int_{\text{wire}} I(s') \left\{ -\frac{\partial}{\partial s} \frac{\partial}{\partial s'} \psi^s + k_0^2 (\hat{s} \bullet \hat{s}') \psi^s \right\} ds'$$
(8)

$$\Delta E_{\psi s} = \int_{\text{wire}} \mathbf{I}(s') \left\{ -\frac{\partial}{\partial s} \frac{\partial}{\partial s'} \Delta \psi^s + k_0^2 \left(\hat{s} \bullet \hat{s}' \right) \Delta \psi^s \right\} ds'$$
(9)

$$E_{\psi} = \int_{\text{wire}} I(s') \frac{\partial}{\partial s} \frac{\partial}{\partial s'} \psi \, ds'$$
(10)

$$\Delta E_{\psi} = \int_{\text{wire}} \mathbf{I}(s') \, \frac{\partial}{\partial s} \frac{\partial}{\partial s'} \, \Delta \psi \, ds' \tag{11}$$

Eq. (8) corresponds to the term obtained for an arbitrarily shaped wire in *free space*. The effect of the dielectric substrate appears in Eqs. (10) and (11). In fact, when $\varepsilon_r = 1$, Eqs. (10) and (11) vanish. Note that Eq. (9) for $\varepsilon_r = 1$ shows the effect of the ground plane.

Let the current be expanded as $I(s') = \sum_{n} I_n J_n(s')$ with $J_n(s')$ being piecewise sinusoidal functions. Then, Eq. (8) for the nth piecewise sinusoidal expansion function, which is defined in a region from s'= s_{n-1} to s_{n+1} , is given as

$$E_{\psi s} = -I_{n} \frac{j30}{\sin k_{0} d} \left[\left(\frac{\mathbf{P}_{n-1}}{\rho_{n-1}} + \frac{\mathbf{P}_{n}}{\rho_{n}} \right) - \left(\mathbf{S}_{n-1} + \mathbf{S}_{n} \right) \right] \bullet \hat{s}$$
(12)

where an equidistant-spacing d $(s_n - s_{n-1} = s_{n+1} - s_n = d)$ is adopted; ρ_j (j = n - 1, n) is the radial distance from the jth wire segment axis to the observation point specified by coordinates (x, y, z); and \mathbf{P}_{n-1} , \mathbf{P}_n , \mathbf{S}_{n-1} , and \mathbf{S}_n are vector quantities. Note that \mathbf{P}_{n-1} is expressed in closed form [1]:

$$\mathbf{P}_{n-1} = [(\hat{\sigma}_{n-1} \bullet \hat{R}_n) \ e^{-jk_0R_n} \cos k_0d \ -(\hat{\sigma}_{n-1} \bullet \hat{R}_{n-1}) \ e^{-jk_0R_{n-1}} - je^{-jk_0R_n} \sin k_0d]\hat{\rho}_{n-1}$$
(13)

where R_j (j = n - 1, n) is the distance from the wire segment end point specified by distance s' = s_j to the

observation point; $\hat{\mathbf{R}}_{j}$ and $\hat{\rho}_{j}$ are the unit vectors in the \mathbf{R}_{j} and ρ_{j} directions, respectively; and $\hat{\sigma}_{j}$ is the unit vector parallel to the jth wire segment axis. The remaining vector quantities \mathbf{P}_{n} , \mathbf{S}_{n-1} , and \mathbf{S}_{n} , which are not presented due to space limitations, are also expressed in closed form.

3. Impedance matrix element Z^{NEW}_{m.n}

After the current I(s') in Eqs. (9)-(11) are expanded using the nth piecewise sinusoidal expansion function, the mth piecewise sinusoidal weighting function $W_m(s) = J_m(s)$ is applied to Eqs. (9)-(11) and Eq. (12) to obtain an impedance matrix element $Z^{NEW}_{m,n}$:

$$Z^{\text{NEW}}_{m,n} = Z^{\psi_{s}}_{m,n} + Z^{\psi}_{m,n} + \Delta Z_{m,n}$$
(14)

where

$$Z^{\psi_{s}}_{m,n} = \frac{1}{\sin k_{0}d} \int_{0}^{d} E_{\psi_{s}} \sin k_{0}\sigma_{m-1} d\sigma_{m-1} + \frac{1}{\sin k_{0}d} \int_{0}^{d} E_{\psi_{s}} \sin k_{0}(d-\sigma_{m}) d\sigma_{m}$$
(15)

$$Z^{\Psi}_{m,n} = \left(\frac{k_0}{\sin k_0 d}\right)^2 \left(g^{\Psi}_{m-1, n-1} + g^{\Psi}_{m-1, n} + g^{\Psi}_{m, n-1} + g^{\Psi}_{m, n}\right)$$
(16)

$$\Delta Z_{m,n} = \left(\frac{k_0}{\sin k_0 d}\right)^2 \left(\Delta g_{m-1, n-1} + \Delta g_{m-1, n} + \Delta g_{m, n-1} + \Delta g_{m, n}\right)$$
(17)

in which

$$g^{\psi_{m-1,n-1}} = \int_{0}^{d} \cos k_0 \sigma_{m-1} \left\{ \int_{0}^{d} \psi_{m-1,n-1} \cos k_0 \sigma'_{n-1} d\sigma'_{n-1} \right\} d\sigma_{m-1}$$
(18)

$$\Delta g_{m-1, n-1} = \int_{0}^{d} \cos k_{0} \sigma_{m-1} \left\{ \int_{0}^{d} (-\Delta \psi^{s}_{m-1, n-1} + \Delta \psi_{m-1, n-1}) \cos k_{0} \sigma'_{n-1} d\sigma'_{n-1} \right\} d\sigma_{m-1}$$
$$+ (\hat{\sigma}_{m-1} \bullet \hat{\sigma}_{n-1}) \int_{0}^{d} \sin k_{0} \sigma_{m-1} \left\{ \int_{0}^{d} \Delta \psi^{s}_{m-1, n-1} \sin k_{0} \sigma'_{n-1} d\sigma'_{n-1} \right\} d\sigma_{m-1}$$
(19)

where new variables σ_{m-1} and σ_m are defined as $\sigma_{m-1} = s - s_{m-1}$ and $\sigma_m = s - s_m$, respectively. Similarly, a new variable σ'_{n-1} is defined as $\sigma'_{n-1} = s' - s_{n-1}$. The subscripts of $\psi_{m-1, n-1}$, $\Delta \psi_{m-1, n-1}$, and $\Delta \psi^s_{m-1, n-1}$ show that these are functions of σ_{m-1} and σ'_{n-1} . The remaining terms $g^{\psi}_{i,j}$ and $\Delta g_{i,j}$ (i = m - 1, m: j = n - 1, n), which are not presented due to space limitations, take forms similar to $g^{\psi}_{m-1, n-1}$ and $\Delta g_{m-1, n-1}$, respectively.

 $Z^{\psi_{s}}{}_{m,n}$ in Eq. (15) and $Z^{\psi}{}_{m,n}$ in Eq. (16) involve single and double integrals, respectively. $\Delta Z_{m,n}$ in Eq. (17) involves triple integrals, because $\Delta g_{i,j}$ (i = m-1, m: j = n-1, n) includes Sommerfeld-type integrals in $\Delta \psi_{i,j}^{s}$ and $\Delta \psi_{i,j}$. Therefore, it can be said that $Z^{NEW}{}_{m,n}$ is simpler than the conventional $Z_{m,n}$, which is composed of $g_{m-1, n-1}$, $g_{m-1, n}$, $g_{m, n-1}$, and $g_{m, n}$ [1], all involving triple integrals.

Using the impedance matrix element Z^{NEW}_{m,n}, a grid array antenna printed on a dielectric substrate

(see Fig. 1) is analyzed. The computational time using $Z^{NEW}_{m,n}$, where interpolation techniques are applied to $\Delta \psi_{i,j}$ and $\Delta \psi_{i,j}^{s}$, is decreased by a factor of 60, compared with that using the conventional $Z_{m,n}$. The current distribution is shown in Fig. 2.

4. Conclusions

A moment method solution to an integral equation for a printed wire is discussed. The tangential electric field along the printed wire is decomposed into four terms: $E_{\psi s}$, $\Delta E_{\psi s}$, E_{ψ} , and ΔE_{ψ} . When piecewise sinusoidal functions are used as expansion functions (basis functions), the first term $E_{\psi s}$ has a closed form. The weighting (testing) process using the same piecewise sinusoidal functions leads to a new impedance matrix element $Z^{NEW}_{m,n}$, which is composed of single-, double-, and triple- integral terms. The computational time using $Z^{NEW}_{m,n}$ is significantly decreased, compared with that using the conventional $Z_{m,n}$.

Reference

[1] H. Nakano, "Chapter 3, Antenna analysis using integral equations" in *Analysis Methods for Electromagnetic Wave Problems, Volume II*, E. Yamashita (Editor), Artech House, Boston, 1996



Fig. 1 Configuration of a grid array antenna



Fig. 2 Current distribution