C–5–3 AN EXACT SOLUTION OF THE FOURTH-ORDER MOMENT EQUATION FOR A PLANE WAVE IN A PARTICULAR CASE

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Abstract — The fourth-order moment equation is exactly solved for a plane wave propagated through the random medium with a particular correlation function. This solution shows precisely the small variance of the irradiance caused by wave-form distortion, and hence gives the range of simple observation of spot dancing.

I. Introduction

The so-called moment equation determines the statistical properties of successively forward-scattered (S F S) waves, and holds[1] under the paraxial approximation and the condition $kl \gg 1$ where k is the wave number in free space and l a correlation length in the medium. So it is a basic equation on analysis of wave propagation through random media. But the moment equation has not been exactly solved except the first- and the second-order moment and the higher moments[2] in the case that the correlation function $B(r,z)$ of the refractive index is approximately expressed as $\sum_0^1 a_{2n}(z)r^{2n}$ where, in general, $B(r,z) = \sum_0^\infty a_{2n}(z)r^{2n}$. The r^2 term in $B(r,z)$ only gives rise to such an idealized spot dancing that each irradiance of S F S waves has the same form , and the arrival position displaces randomly. Other terms r^{2n} $(n\neq 0,1)$ yield the random deformation of the irradiance[3].

In the present paper, we solve exactly the fourth-order moment equation for a plane wave in the case $B(r,z) = \sum_0^2 a_{2n}(z)r^{2n}$, by using well-known techniques. This solution affords the condition for the aforementioned spot dancing to be observable, a criterion for examining the accuracy of approximate solutions in a more general case, and so on.

II. Formulation

Let $u(\mathbf{r},z)$ represent an S F S wave in the medium with the refractive index $n = n_0(1 + \delta\varepsilon(\mathbf{r},z))^{1/2}$ where n_0 is a constant and $\delta\varepsilon(\mathbf{r},z)$ a Gaussian random function satisfying

$$\langle\delta\varepsilon(\mathbf{r},z)\rangle = 0 \qquad \langle\delta\varepsilon(\mathbf{r}_1,z_1)\delta\varepsilon(\mathbf{r}_2,z_2)\rangle = B(\mathbf{r}_1 - \mathbf{r}_2, z_1 - z_2)$$

in which the angular brackets denote the ensemble average. The fourth-order moment of the S F S wave is defined in a $\mathbf{r}$-plane, z = constant, as

$$M_{22}(\mathbf{s}_1,\mathbf{s}_2,\mathbf{t}_1,\mathbf{t}_2,z) = \langle\prod_{i=1}^{2} u(\mathbf{s}_i,z)\prod_{i=1}^{2} u^*(\mathbf{t}_i,z)\rangle$$

where the asterisk denotes complex conjugate.

Changing the variables $\mathbf{s}_i$ and $\mathbf{t}_i$ into $\mathbf{r}_{-i}$ and $\mathbf{r}_i$:

$$\mathbf{r}_{-i} = \mathbf{s}_i - \mathbf{t}_i \qquad \mathbf{r}_1 = (\mathbf{s}_1 - \mathbf{s}_2 + \mathbf{t}_1 - \mathbf{t}_2)/2 \qquad \mathbf{r}_2 = (\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{t}_1 + \mathbf{t}_2)/4$$

and using D(r,z) defined by

$$D(\mathbf{r},z) = 2(B(0,z) - B(\mathbf{r},z)) ,$$

we can write the fourth-order moment equation as

$$\{\frac{\partial}{\partial z} - j\frac{1}{k}[(\nabla_{r_1} + \frac{1}{2}\nabla_{r_2})\cdot\nabla_{r_{-1}} + (-\nabla_{r_1} + \frac{1}{2}\nabla_{r_2})\cdot\nabla_{r_{-2}}]\} M_{22}(\mathbf{r}_i,\mathbf{r}_{-i},z)$$

$$= [-\frac{k^2}{4}\int_0^z dz_1(D_1(\mathbf{r}_{-i},z_1) + D_2(\mathbf{r}_1,\mathbf{r}_{-i},z_1))]\, M_{22}(\mathbf{r}_i,\mathbf{r}_{-i},z)$$

$$M_{22}(\mathbf{r}_i,\mathbf{r}_{-i},0) = M_{22}^{in}(\mathbf{r}_i,\mathbf{r}_{-i},0) . \quad (1)$$

Here, the operators ∇_{r_i}, $\nabla_{r_{-i}}$ represent the two-dimensional gradient, and

$$D_1(\mathbf{r}_{-i},z) = D(\mathbf{r}_{-1},z) + D(\mathbf{r}_{-2},z)$$

$$D_2(\mathbf{r}_1,\mathbf{r}_{-i},z) = D(\mathbf{r}_1 + \frac{\mathbf{r}_{-1}+\mathbf{r}_{-2}}{2},z) + D(\mathbf{r}_1 - \frac{\mathbf{r}_{-1}+\mathbf{r}_{-2}}{2},z)$$

$$- D(\mathbf{r}_1 + \frac{\mathbf{r}_{-1}-\mathbf{r}_{-2}}{2},z) - D(\mathbf{r}_1 - \frac{\mathbf{r}_{-1}-\mathbf{r}_{-2}}{2},z)$$

$$M_{22}^{in}(\mathbf{r}_i,\mathbf{r}_{-i},z) = \prod_{i=1}^{2} u_{in}(\mathbf{s}_i,z)\prod_{i=1}^{2} u_{in}^{*}(\mathbf{t}_i,z)$$

where $u_{in}(\mathbf{r},z)$ is a wave function in the non-random ($\delta\epsilon \equiv 0$) medium.

Let us define the Fourier transform and its inverse as

$$\hat{f}(\boldsymbol{\kappa}) = \int f(\mathbf{r})\exp(-j\boldsymbol{\kappa}\cdot\mathbf{r})\, d\mathbf{r} \qquad f(\mathbf{r}) = (2\pi)^{-2}\int \hat{f}(\boldsymbol{\kappa})\exp(j\mathbf{r}\cdot\boldsymbol{\kappa})\, d\boldsymbol{\kappa} .$$

Then, taking the Fourier transform of (1) with respect to $\mathbf{r}_i$, and putting

$$\mathbf{r}_{-i} = \boldsymbol{\rho}_i + z[(-1)^{i+1}\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2/2]/k ,$$

we get

$$\frac{\partial}{\partial z}\hat{M}_{22}(\boldsymbol{\kappa}_i,\boldsymbol{\rho}_i+\frac{z}{k}((-1)^{i+1}\boldsymbol{\kappa}_1+\frac{1}{2}\boldsymbol{\kappa}_2),z) = \{-\frac{k^2}{4}\int_0^z dz_1[D_1(\boldsymbol{\rho}_i+\frac{z}{k}((-1)^{i+1}\boldsymbol{\kappa}_1+\frac{1}{2}\boldsymbol{\kappa}_2),z_1)$$

$$+ \hat{D}_2(j\nabla_{\kappa_1},\boldsymbol{\rho}_i+\frac{z}{k}((-1)^{i+1}\boldsymbol{\kappa}_1+\frac{1}{2}\boldsymbol{\kappa}_2),z_1)]\}\, \hat{M}_{22}$$

$$\hat{M}_{22}(\boldsymbol{\kappa}_i,\mathbf{r}_{-i},0) = \hat{M}_{22}^{in}(\boldsymbol{\kappa}_i,\mathbf{r}_{-i},z) . \quad (2)$$

The solution of the above equation is written as

$$\hat{M}_{22}(\boldsymbol{\kappa}_i,\mathbf{r}_{-i},z) = \sum_{n=0}^{\infty}\hat{M}_{22}^{(n)}(\boldsymbol{\kappa}_i,\mathbf{r}_{-i},z)$$

$$\hat{M}_{22}^{(0)} = \hat{M}_{22}^{in}(\boldsymbol{\kappa}_i,\mathbf{r}_{-i},z)A(z)$$

$$\hat{M}_{22}^{(n)} = A(z)\int_0^z dz_1\int_0^{z_1} dz_2\cdots\int_0^{z_{n-1}} dz_n[(A^{-1}(z_1)D_s(z_1)A(z_1))(A^{-1}(z_2)D_s(z_2)A(z_2))$$

$$\cdots(A^{-1}(z_n)D_s(z_n)A(z_n))]\ M_{22}^{in}(\kappa_i,r_{-i},z) \tag{3}$$

in which

$$D_s(z_m) = -\frac{k^2}{4}\int_0^{z_m} dz'\ \hat{D}_2(j\nabla_{\kappa_1}, r_{-i} - \frac{z-z_m}{k}((-1)^{i+1}\kappa_1 + \frac{1}{2}\kappa_2), z') \tag{4}$$

$$A(z_m) = \exp[-\frac{k^2}{4}\int_0^{z_m} dz' \int_0^{z'} dz''\ D_1(r_{-i} - \frac{z-z'}{k}((-1)^{i+1}\kappa_1 + \frac{1}{2}\kappa_2), z'')]\ . \tag{5}$$

Noting that ∇_{κ_1} in D_s operates only $\hat{M}_{22}$ in (2) and that $D(r,z)$, in general, is expressed as

$$D(r,z) = \sum_{n=0}^{\infty} b_{2n}(z) r^{2(n+1)} \qquad b_{2n}(z) = [\nabla_r^{2(n+1)} D(r,z)]_{r=0} / [(2n+2)!!\ ]^2, \tag{6}$$

we have

$$D_s(z_m) = 2\sum_{n=0}^{\infty} B_{2n}(z_m) \sum_{g=1}^{n+1} \binom{2(n+1)}{2q} [(\frac{r_{-1}+r_{-2}}{2} - \frac{z-z_m}{2k}\kappa_2)^{2q}$$
$$- (\frac{r_{-1}-r_{-2}}{2} - \frac{z-z_m}{k}\kappa_1)^{2q}] \cdot (j\nabla_{\kappa_1})^{2(n+1-q)} \tag{4'}$$

$$A(z_m) = \exp\{\sum_{n=0}^{\infty}\int_0^{z_m} dz' B_{2n}(z') [(r_{-1} - \frac{z-z'}{k}(\kappa_1 + \frac{1}{2}\kappa_2))^{2(n+1)}$$
$$+ (r_{-2} + \frac{z-z'}{k}(\kappa_1 - \frac{1}{2}\kappa_2))^{2(n+1)}] \tag{5'}$$

where

$$B_{2n}(z) = -\frac{k^2}{4}\int_0^{z} b_{2n}(z') dz'\ .$$

III. Irradiance Variance of a Plane Wave in a Particular Case

Consider the plane wave propagation: $u_{in}(r,z) = \exp(jkz)$. Then we have

$$M_{22}^{in}(\kappa_i, r_{-i}, z) = (2\pi)^4 \delta(\kappa_1)\delta(\kappa_2)\ .$$

We are concerned here with the irradiance of the wave and hence put $r_{-1} = r_{-2} = 0$. In the case

$$D(r,z) = \sum_{n=0}^{1} b_{2n}(z) r^{2(n+1)} \tag{7}$$

the inverse transform of (3) becomes

$$M_{22}^{(n)} = \int d\kappa\ \exp(j r_1 \cdot \kappa_1)[A(z)\int_0^{z} dz_1 \cdots \int_0^{z_{n-1}} dz_n ((A^{-1}(z_1)D_s(z_1)A(z_1))\cdots$$
$$\cdot (A^{-1}(z_n)D_s(z_n)A(z_n))\delta(\kappa_1)] \tag{8}$$

where $A(z_m)$ and $D_s(z_m)$ respectively are

$$D_s(z_m) = -2\left(\frac{z-z_m}{k}\kappa_1\right)^2\{B_0(z_m) + B_2(z_m)[(\frac{z-z_m}{k}\kappa_1)^2 - 6\nabla^2_{\kappa_1}]\}$$

$$A(z_m) = \exp[\sum_{n=0}^{1}\int_0^{z_m} B_{2n}(z')(\frac{z-z'}{k}\kappa_1)^{2(n+1)}dz'] .$$

The application of

$$\int f_1(\kappa)\nabla^2_\kappa f_2(\kappa)\delta(\kappa)d\kappa = f_2(0)[\nabla^2_\kappa f_1(\kappa)]_{\kappa=0}$$

to (8) leads to the fourth-order moment:

$$M_{22}(r_1,z) = \sum_{n=0}^{\infty} M_{22}^{(n)}(r_1,z)$$

$$= \sum_{n=0}^{\infty}\int_0^z dz_1\cdots\int_0^{z_{n-1}} dz_n\{[-12(z-z_1)^2\int_0^{z_1} b_2(z_1')dz_1']\cdots[-12(z-z_n)^2\int_0^{z_n} b_2(z_n')dz_n']\}$$

$$= \exp[g(z)] \qquad g(z) = -4\int_0^z (z-z')^3 b_2(z')dz' . \tag{9}$$

Then, the irradiance variance σ^2 is given by

$$\sigma^2 = \exp[g(z)] - 1 .$$

For example, if

$D(r,z) = 2B(0,0)[1-\exp(-(r/l)^2)]\exp(-(z/l)^2)$

we have

$$b_2(z) = -l^{-4}B(0,0)\exp(-(z/l)^2).$$

Hence

$$g(z) = 4l^{-4}B(0,0)\int_0^z (z-z')^3\exp(-(z'/l)^2)dz'$$

$$\simeq 2\sqrt{\pi}\, B(0,0)(z/l)^3 \quad \text{for } z \gg l.$$

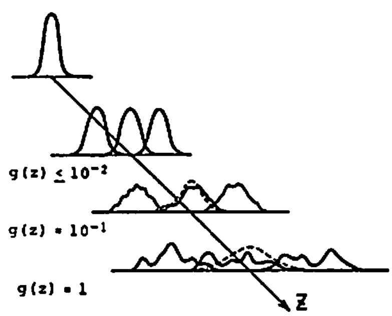

Fig. 1. Wave Beams Propagated through Random Media

IV. Conclusions

We have already described the effects of the r^2 term in $D(r,z)$[3] and may easily obtain the properties of $M_{22}^{(1)}(r_1,z)$ in the general case: $D(r,z) = \sum^{\infty} b_{2n}(z)r^{2(n+1)}$. When these are added to a new result of the present analysis, the following conclusions can be drawn.

(1) If $g(z) \ll 1$, the irradiance variance of a plane wave is negligible, and hence it is considered that random displacement of the arrival positions (spot dancing) only may exist. Here a problem concerning spot dancing has been solved. In $g(z) \simeq 10^{-1}$, the small variance of the irradiance occurs even for a plane wave. Then a wave form is appreciately distorted. If $g(z) \simeq 1$, a wave is largely deformed, and the irradiance variance should be discussed in the general case. This mention is shown in Fig. 1 .

(2) For a plane wave, the correlation of the irradiance depends not on the r^2 and r^4 terms in $D(r,z)$ but on the $r^{2(n+1)}$, n=2,3,$\cdots$, terms.

(3) The exact solution derived in III may be used as a criterion for testing the accuracy of the approximate fourth-order moments in a more general case.

References [1]M.Tateiba,Mem.Fac.Eng.Kyushu Univ.,33,129('74). [2]K.Furutsu, J.Opt.Soc.Amer.,62,240('72). [3]M.Tateiba,IEEE Trans.,AP-23,493('75).