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ON THE MODIFIED RAYLEIGH HYPOTHESIS AND THE MODE-MATCHING METHOD

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Introduction

The method of mode-matching is a fundamental technique for solving numerically the boundary value problems for the Helmholtz equation. In the applications to scatterers of arbitrary shapes, this method has the basis in the Rayleigh hypothesis. There have appeared recently several papers[1)] on the legitimacy of "the Rayleigh hypothesis and the method of mode-matching", and most of them have derived the disadvantageous conclusions for the hypothesis and the method.

In the exterior or interior problem, the Rayleigh hypothesis closely relates with the completeness of the modal expansion. In this paper, a new approach to the concept of the modal expansion and the modified Rayleigh hypothesis is presented. Instead of the "infinite" series of modal functions, a sequence whose elements are "finite" series of them is constructed so that it converges to the true field in the whole exterior or interior domain. This approach may be explained by the "correspondence principle" between the wave theory and the function theory. The algorithm of the mode-matching method and the improved point-matching method are presented.

For brevity, the argument is restricted to the Dirichlet problem for the two dimensional scalar fields in an exterior domain. The results of numerical analyses for a cylinder with an arbitrary geometrical cross section is shown as examples of the new method. It is assumed that the time dependence is the usual form $\exp(j\omega t)$.

The correspondence principle

The Helmholtz equation, with the wave number k, is

$$(\partial^2/\partial x^2+\partial^2/\partial y^2)\Psi(P) + k^2\Psi(P) = 0$$

where P denotes a point with cartesian coordinates (x, y) or cylindrical coordinates (ρ, θ), and the origin is set inside the scatterer. The wave function $\Psi(P)$ satisfies the radiation condition at the infinity.

Let $H_n^{(2)}(k\rho)$ denote the Hankel function of the second kind, then a system of modal functions of the wave theory can be defined by

$$\phi_n(P) = H_n^{(2)}(k\rho)e^{jn\theta} \qquad (n=0,\pm1,\pm2,\cdots)$$

The correspondence principle states that the gradients of modal functions

$$\{\nabla\phi_{|n|}(P)\}$$

play the same roles as $\{1/Z^{n+1}\}$ of the function theory, where Z is x+iy. By analogy with Runge's theorem in the function theory, any wave function can be uniformly approximated, in the exterior domain, by a finite series of the form

$$\Lambda_N(P) = \sum_{n=-N}^{N} b_n(N)\phi_n(P) \qquad (1)$$

and Ψ is precisely represented as the limiting wave function of the sequence $\{\Lambda_N\}$. In Eq.(1), the coefficients $\{b_n(N)\}$ depend on the truncated number N.

These arguments show a modification of the Rayleigh hypothesis and a new concept for the completeness of modal expansions.[2,3,4)]

The mode-matching method

In the application to the boundary

value problem, the coefficients $\{b_n(N)\}$ in Λ_N can be obtained from the mode-matching technique on the boundary.

We consider the Dirichlet problem: to find a wave function which satisfies the boundary condition

$$\Psi(Q) = f(Q), \quad Q\in L$$

where Q denotes a point on the boundary L. The mode-matching method is formulated in the following procedure. Let the normalized mean square error be defined by

$$\Omega_N = \int_L |\Lambda_N(Q)-f(Q)|^2 ds \Big/ \int_L |f(Q)|^2 ds, \tag{2}$$

Minimizing Ω_N, we can obtain the suitable set of coefficients $\{b_n(N)\}$. For the computer-aided calculation, the boundary is divided into 2M+1(≧2N+1) subsections and the integral of Eq.(2) is approximated by the summation of (2M+1) terms, that is,

$$\Omega(N, M) = \frac{\sum_{i=-M}^{M} |\Lambda_N(Q_i)-f(Q_i)|^2 \Delta s_i}{\sum_{i=-M}^{M} |f(Q_i)|^2 \Delta s_i}$$

where Δs_i is the length of the i-th subsection. The minimization of $\Omega(N, M)$ is obtained by means of least square. When M is equal to N, we get the point-matching method, but this method generally leads to defective solutions. If M is larger than 2N, we have practically acceptable solutions for the sufficiently large N. This procedure may improve the point-matching method.

Example

Numerical results are shown for the scattering of a cylinder with a geometrical cross section

$$\rho' = a(1+\delta\cos 3\theta')$$

The incident wave is a plane wave polarized parrallel to the generator of cylinder and its direction of propagation is denoted by the angle ϕ_0. In this example, if M is larger than 2N, we have favorable results for a sufficient large N. E_{opt} indicates the error of the numerical result for the optical theorem, which states the relation between the total cross section and the forward scattering amplitude. In Fig. 2, the curves of $\Omega(N, 2N)$ and $E_{opt}(N, 2N)$ show the results of the improved point-matching, and $E_{opt}(N, N)$ is calculated by the usual point-matching method.

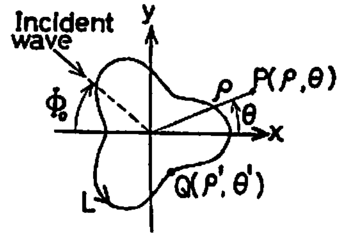

Fig. 1 Geometry of a scatterer

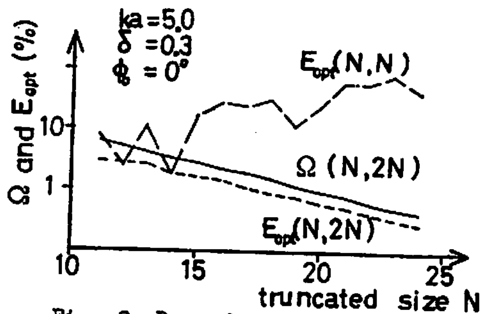

Fig. 2 Dependence of the errors on the truncated size N

References

1) L. Lewin: "On Inadequacy of Discreet mode-matching Techniques in Some Waveguide Discontinuity Problems" IEEE Trans. Vol.MTT-18, No.7, pp.364-372, 1970

2) A. P. Calderón: "The Multipole Expansion of Radiation Fields" J. of Rat. Mech. Anal., Vol.3, pp.523-534, 1954

3) I. N. Vekua: "New Methods for Solving Elliptic Equations" D. E. Brown and A. B. Tayler, trans., Amsterdam, North-Holland Publishing Co. Chap.2, 1967

4) K. Yasuura and T. Itakura: "Approximate Method for Wave Functions" I, II, III, Kyushu Univ. Tech. Rep., Vol.38, pp.72-77, Vol.38, pp.378-385, Vol.39, pp.51-56, 1965-1966 (in Japanese)