A NEW "EXACT" SOLUTION FOR DISCRETE INVERSE PROBLEMS

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Introduction

This paper presents a new method for solving inverse scattering and inverse source problems. The method is "exact", in that no approximations regarding the characteristics of the source, scatterer, or scattering process are made, save possibly for a very weak restriction on the functional form of the frequency or time dependence of the scatter or the source. The approach is equally valid for the scalar and vector cases of a wide variety of inverse problems, including those which can be formulated using the Helmholtz, Schrodinger, and acoustic wave equations. However, it is derived from the start, and inherently applies to, the discrete case: problems in which the scattering data (for the inverse scattering problem) or the radiated fields (for the inverse source problem) are known only at discrete positions in space, in which the solution is sought only for discrete positions in space, and in which the data and solutions are discrete functions of frequency or time. This approach is well-suited for most "real-world" inverse problems, where data are usually inherently only known at discrete points because of the measurement process. However, it can obviously never yield a totally "exact" solution, since information about the source or scatterer between the solution points is indeterminable (hence the use of quotation marks around exact).

Because of space limitations, this paper concentrates on deriving the results as succinctly as possible. Familiarity with the literature on inverse problems is assumed. Although scalar notation is used for simplicity of presentation and to minimize the number of equations, all of the results apply to the full vector case. The results have been extended to anisotropic scatterers or sources in a straightforward manner. This solution was first obtained by the author in January of 1983. Although the author is unaware of any similar approaches, the solution is simple enough that it would not be surprising to find that others have independently arrived at the same approach.

Derivation of the solution

The problem to be solved can be stated mathematically as follows. Let $\phi(\mathbf{x},\omega)$ denote the total field (where **bold** quantities denote spatial vectors). The data in the problem consist of measurements of ϕ over some surface enclosing the source or scatterer. Let $\rho(\mathbf{x},\omega)$ denote the source. For the inverse source problem, this is the unknown. For the inverse scattering problem, this source term is related to the unknown scatterer's constituent properties (*e.g.*, the permeability and permitivity, or the complex refractive index, $n(\mathbf{x},\omega)$). The wave equation is given by

$$\nabla^2\phi(\mathbf{x},\omega) + k^2\phi(\mathbf{x},\omega) = -\rho(\mathbf{x},\omega) \qquad \text{where } k = 2\pi/\lambda \qquad (1)$$

and λ is the free-space wavelength. The consituative equation is

$$\rho(\mathbf{x},\omega) = k^2[n^2(\mathbf{x},\omega) - 1]\ \phi(\mathbf{x},\omega) \quad (2)$$

The free-space Green's function, G, satisfies

$$\nabla^2 G(\mathbf{x},\omega) + k^2 G(\mathbf{x},\omega) = -\delta(\mathbf{x},\omega) \quad (3)$$

A simplification will be made *for now only*: let ρ be assumed independent of ω. Then Equation (1) becomes

$$\nabla^2 \phi(\mathbf{x},\omega) + k^2 \phi(\mathbf{x},\omega) = -\rho(\mathbf{x}) \quad (4)$$

For the discrete solution, the source can be written as

$$\rho(\mathbf{x},\omega) = \sum_{p=1}^{P} \sum_{q=1}^{Q} A_{pq}\delta(x-x_p,z-z_q) \quad (5)$$

Note that Equation (5) applies to the two-dimensional, scalar case, and the two-dimensional case will be used to illustrate the rest of the derivation. This is done only to minimize the number of indices in the derivation which follows; the same approach works for one, two, or three dimensions. Note also that the As in Equation 5 are complex. Equation 5 is the "trick" in this solution: the source term is discretized *before* the solution is developed further and, in particular, *before* substituting the source term into any expression for the fields.

The solution to the wave equation (1) is

$$\phi(x,z,\omega) = \int dV'\ G(x-x',z-z',\omega)\ \rho(x',z') \quad (6)$$

For the vector electromagnetic case, the Stratton-Chu equation would be used in place of Equation (6). Equation (5) is substituted into Equation (6) to yield

$$\phi(x,z,\omega) = \iint dx'dz'\ G(x-x',z-z',\omega) \sum_{p=1}^{P} \sum_{q=1}^{Q} A_{pq}\delta(x'-x_p,z'-z_q) \quad (7)$$

$$\phi(x,z,\omega) = \sum_{p=1}^{P} \sum_{q=1}^{Q} A_{pq}G(x-x_p,z-z_q,\omega) \quad (8)$$

Equation (8) is obtained from Equation (7) by interchanging the order of summation and integration (invoking Fubini), and integrating over the delta distributions. Equation (8) is the solution for the two-dimensional case with a frequency-independent source or scatterer. For this case (and without loss of generality), the measurement surface can be taken to be the plane z =

0. Then Equation (8) becomes

$$\phi(x_\ell,z,\omega_r) = \sum_{p=1}^{P} \sum_{q=1}^{Q} A_{pq} G(x-x_p,z-z_q,\omega_r), \quad \ell = 1,...,M; \; r = 1,...,R \tag{9}$$

Equation (9) must be solved by matrix inversion. The form of Equation (9) helps to make explicit the relationships among the number of required data and the number of available solution points. For instance, in the notation of Equation (9), it is necessary that $M \geq P$ and $R \geq Q$. If $M < P$, then R must be proportionally $> Q$ (and the response of the scatterer must be sufficiently different with frequency [and the assumption of frequency independence must be false in this case - see the comments below]), or the solution will be underdetermined. Similarly, if both $M > P$ and $R > Q$, the solution is overdetermined. Note that oversampling (*i.e.*, recording data at more points than are sought in the solution) in x could be substituted for frequency information. Equation (9) also identifies a common source of ill-posed inverse problems: insufficient *independent* data points. Since Equation (9) can be interpreted as a set of simultaneous linear equations, the standard requirements on the independence of this set of equations can immediately be interpreted in terms of the independence of the scattering data (for example).

The source term, ρ, was taken to be independent of frequency in Equation (5) to simplify the notation. There are several ways dispersive sources (or scatterers) can be accommodated. The most straightforward method is to substitute the following for Equation (5):

$$\rho(x,\omega) = \sum_{u=1}^{U} \sum_{p=1}^{P} \sum_{q=1}^{Q} A_{pqu} \delta(x-x_p,z-z_q,\omega_u) \tag{10}$$

At "worst", the linearity of the wave equation results in one additional summation in the solution in this case.

Note that for the inverse source problem, ϕ is known from the measured data, G is the (known) free-space Green's function, and solution of Equation (9) yields the A_{pq}s (or, if Equation (10) is used, the A_{pqu}s). Substitution into Equation (5), which is trivial (or Equation (10)), gives the unknown source values. For the inverse scattering problem, these same steps are followed to obtain the values of ρ, and these values of ρ and the measured values of ϕ are used in Equation (2) to solve for the refractive index, n.

Properties of the approach, and comments

It is worth noting the following properties of the solution approach:

1. As stated in the Introduction, the solution is "exact" in the sense that there are no approximations made regarding the physics of the problem. The only approximations are directly related to the discretization.

2. While Equation (5) is a particular form of the consituative equation, the derivation is independent of the form of this equation. The approach is thus applicable to a broad class of inverse scattering problems.

3. The method is efficient. The primary computational cost is in the matrix inversion necessary to solve Equation (9). The matrix involved is a discretized form of the free-space Green's function. It is therefore a Toeplitz or block-Toeplitz matrix, for which several efficient, robust inversion methods exist. The values of this matrix also fall off rapidly away from the diagonal. While this property can be exploited for improved efficiency and accuracy, it can also be the source of a very large condition number (particularly if the distance from source or scatterer to measurement point is large in units of wavelength), with its attendant problems.

4. It is only necessary to invert the matrix once for a given geometry.

5. Although it is beyond the scope of this paper, it can be shown that regardless of the dimensionality of the problem, Equation (9) can always be written in terms of one- and two-dimensional matrices by proper arrangement of terms.

6. The matrix to be inverted is nonsingular: The free-space Green's function is bounded and has no zeros (away from the origin) in the complex plane. This should be contrasted with other formulations, which, when discretized, can yield matrix equations to be inverted similar to Equation (9), but involving only the imaginary part of the free-space Green's function (*e.g.*, the "Porter-Bojarski" integral equation).

7. The measured field data enter the formulation in a very straightforward manner. In contrast to some other approaches, there is no requirement to evaluate an auxiliary function of the data (with attendant problems) before solving the inverse problem.

8. The required trade-offs among sampling (or over-sampling) in the spatial dimensions and in frequency are quite evident: Depending on the data, the set of linear simultaneous equations is under-determined, exactly determined, or over-determined. Furthermore, evaluation of the eigenvalues of the problem can yield insight into the dominant radiating or scattering properties of the unknown.

The form of Equation 5 (or Equation (10)) is somewhat arbitrary. A solution with similar properties results from the use of a variety of expansions, and, to some extent, this provides a way to both take advantage of *a priori* information regarding the source or the scatterer and/or to "tailor" the computational properties of the solution. There is an obvious similarity to the Method of Moments (MoM), both in the use of "weights" (the As) and "sampling functions" (chosen to be δ-distributions in the above) [interestingly, this similarity was noted months after the solution was obtained]. Just as in MoM formulations, alternative weights and sampling functions can be used. Even if the chosen sampling functions do not permit the integration of Equation (7) to be carried out analytically, an efficient numerical solution is still possible. One method is to use a weighted Gaussian integration formula in which the weighting function is the same as the expansion sampling function. The relationship between the inverse solution approach presented here and other approaches, including the relationship to the Method of Moments, is the topic of another paper.