

THEORETICAL LIMITATION OF FADING REDUCTION
BY SPACE OR FREQUENCY DIVERSITY

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Abstract: The theoretical relations among Rayleigh, normal and dB-normal distributions are investigated, and is introduced a condition which leads short term median values of Rayleigh fading to dB-normal distribution. From the relation between Rayleigh and dB-normal distributions, the limitations of fading reduction by space or frequency diversity are induced, which are 1.8, 6.7, 13.0, 19.8 and 27.1 dB at cumulative distribution values of 10, 1, 0.1, 0.01 and 0.001 %, respectively. These numerical values will be valid for the diversity system, which keeps the median value constant, and which reduces Rayleigh fading by averaging the received signals of independent levels with respect to the space or frequency. The calculated values are compared with experimental values of a frequency-hopping multilevel FSK system.

1. Introduction: It is well known that a multipath effect of radio wave propagation causes Rayleigh fading, and that the short term median value of Rayleigh fading has dB-normal distribution. The reason for the Rayleigh fading is already explained mathematically,⁽¹⁾ but the dB-normal distribution is not explained yet. Nor is explained either the relation between distributions of a sea wave, whose instantaneous amplitude and wave height have Rayleigh and dB-normal distributions, respectively. An example of propagation circumstances is introduced to induce dB-normal distribution from Rayleigh fading. The theoretical limit to the reduction of Rayleigh fading by space or frequency diversity is induced, considering the relations between Rayleigh and dB-normal distributions, and between time and space averages on the ergodic hypothesis. The result is compared with that of the experiment⁽²⁾ which used a frequency-hopping spread spectrum technique to reduce fading, taking advantage of the frequency diversity effect.

2. Outline of Rayleigh, Normal and dB-normal Distributions: When a probability density function $p(z)$ which is defined by n -dimensional variable z within the range of $-\infty < z < \infty$, and continuous and smooth, has only one extreme value which is maximum at the mean value m , and $p(z-m) = p(m-z)$, then $p(z)$ is proven to be a normal probability density function expressed as follows:

$$p(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-m)^2/2\sigma^2}$$

where $m = \int_{-\infty}^{\infty} z p(z) dz, \quad \sigma^2 = \int_{-\infty}^{\infty} (z-m)^2 p(z) dz = \int_{-\infty}^{\infty} z^2 p(z) dz - m^2$

A radio wave, which is composed of the amplitude and phase, is expressed by the point on the Gauss plane. Many radio waves with random amplitudes and phases can be considered to compose one wave, and the corresponding point on the Gauss plane will have two-dimensional normal distribution with the same standard deviation, because there is in general no reason why the above-mentioned necessary and sufficient condition of normal distribution is not satisfied, and why the two standard deviations come to be different. Then, we can get the relation as follows:

$$\iint_{-\infty}^{\infty} p(x) p(y) dx dy = \iint_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} dx dy = \int_0^{\infty} \int_0^{2\pi} \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r d\theta dr$$

The above formulae show that the two-dimensional normal distribution averaged with respect to the phase comes to be Rayleigh distribution. Making the gradients of Rayleigh, normal and dB-normal cumulative distributions at 50 % values equal, and using the median value of Rayleigh distribution as the measuring unit of arguments, we can express Rayleigh, normal and dB-normal distributions by the following formulae, where p and P represent the probability density function and cumulative distribution function, and the subscript R, n and d represent the values of Rayleigh, normal and dB-normal distributions, respectively. These are drawn in dB scales in Figs.1 and 2.

$$p_R dr = 2 \ln 2 r e^{-(\ln 2) r^2} dr \tag{1}$$

$$p_n dr = \ln 2 e^{-\pi(\ln 2)^2 (r-1)^2} dr \tag{2}$$

$$p_d dr = \frac{\ln 2}{r} e^{-\pi(\ln 2)^2 (\ln r)^2} dr = \ln 2 e^{-\pi(\ln 2)^2 x^2} dx \tag{3}$$

where $x = \ln r$

$$P_R = \int_0^r p_R dr = 1 - e^{-(\ln 2) r^2} \tag{4}$$

$$P_n = \int_{-\infty}^r p_n dr \tag{5}$$

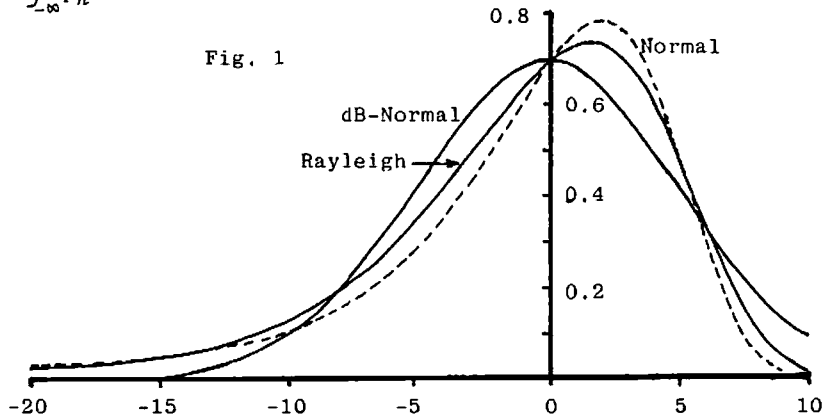


Fig. 1

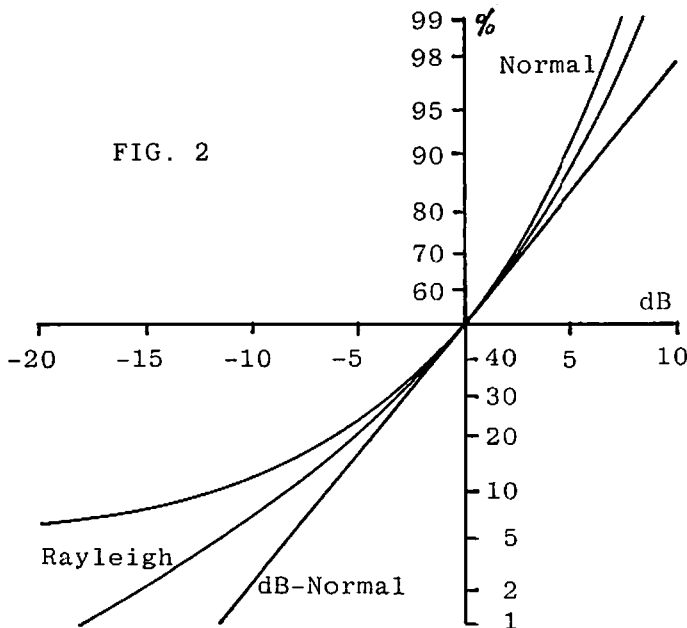


FIG. 2

Fig.1 Density Functions of Rayleigh, Normal and DB-Normal Distributions, Which Correspond to Fig.2.

Fig.2 Cumulative Distributions of Rayleigh, Normal and DB-Normal Density Functions, Whose Gradients at 50 % are Equal.

In Both Figs., Unit of Abscissa is Median Value of Rayleigh Distribution.

$$P_d = \int_0^r p_d dr = \ln 2 \int_{-\infty}^x e^{-\pi(\ln 2)^2 x^2} dx \quad (6)$$

3. Relation Between Rayleigh and dB-Normal Distributions: As mentioned in the previous chapter, the short term median values of Rayleigh fading have dB-normal distribution, but it is impossible to induce mathematically dB-normal distribution from Rayleigh fading by taking its short term average, because Rayleigh fading has no restriction to be dB-normal on the short term distribution. The reason why the short term average has dB-normal distribution should be found in the phenomena which affect the propagation and which make the mean value change. In the following, an explanation is shown by knife edge diffraction and atmospheric refraction.

The diffraction loss L in dB is, on a usual propagation path, proportional to the positive diffraction angle τ . Namely,

$$L \text{ dB} \propto \tau \quad \text{where } \tau > 0 \quad (7)$$

The above relation will be effective enough to explain dB-normal distribution in a mobile communication service system, because the diffraction angles can be considered to distribute normally in the Rayleigh fading region.

The atmospheric refraction $\Delta\tau$ shown in Fig.3 is expressed as follows:

$$\Delta\tau = \int_1^{n_s} \frac{\cot \theta}{n} dn \quad (8)$$

where θ : elevation angle on the path

n_s : refractive index at the receiving point.

On the assumption that initial value of θ is zero where the initial value of refractive index n is unity, Eq.(8) is easily integrated, because

$$n \cos \theta = n_0 \cos \theta_0 = 1 \quad \text{where } n_0 \text{ and } \theta_0 \text{ are initial values of } n \text{ and } \theta.$$

i.e. $\cot \theta = 1/\sqrt{n^2 - 1}$

$$\text{Therefore, } \Delta\tau = \int_1^{n_s} \frac{dn}{n\sqrt{n^2-1}} = \left[-\sin^{-1} \frac{1}{n} \right]_1^{n_s} = \frac{\pi}{2} - \sin^{-1} \frac{1}{n_s}$$

$$\text{Developing the arc sine, } n_s = 1 + \frac{\Delta\tau^2}{2}$$

The refractive index of the atmosphere is usually expressed as follows:

$$n_s = 1 + N_s$$

$$N_s = \left\{ \frac{77.6 P_s}{273 + t_e} + \frac{373256 P_e}{(273 + t_e)^2} \right\} \times 10^{-6} \quad (9)$$

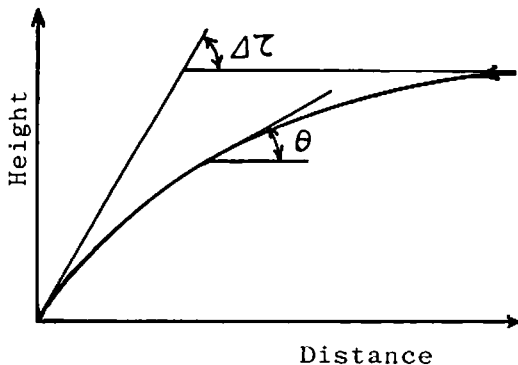


Fig.3 Variation of Diffraction Angle τ Caused by Atmospheric Refraction.

where p_s : pressure in mb, p_e : vapour pressure in mb, t_e : temperature in $^{\circ}\text{C}$

The partial derivatives of (9) with respect to p_s , p_e and t_e show that N_s distributes in the same way as p_s , p_e and t_e . The short term mean values of p_s , p_e and t_e are meteorologically proven to distribute normally. Then, Eqs. (7) - (9) show that a fixed communication service system has dB-normal distribution of the short term mean level variation caused by atmospheric variation in the Rayleigh fading region.

4. Theoretical Limitation of Fading Reduction by Space or Frequency Diversity

In the previous chapters, it is explained that the multipath effect causes Rayleigh fading, and the short term average has dB-normal distribution. Namely, two-dimensional normal distribution averaged with respect to the phase comes to be Rayleigh distribution, and the Rayleigh fading averaged with respect to a term which is relatively short but longer than the fading period comes to be dB-normal distribution. The ergodic hypothesis mentions that the time average is statistically equal to the space average, and this hypothesis is considered to be valid for almost all physical phenomena. Applying the ergodic hypothesis to the Rayleigh fading, the Rayleigh fading averaged with respect to time is equal to the fading averaged with respect to space. Namely the space diversity, which averages many signals of independent Rayleigh fading along different propagation paths, produces the signal of dB-normal distribution.

In a radio wave link which adopts a frequency hopping spread spectrum technique to reduce the fading, the mean signal level is obtained by averaging the spreaded spectrum in which individual fading is considered to be independent. In this case, averaging frequencies means averaging the fading effects on the propagation path, because the different frequencies with independent fading is considered to be the different frequencies with different propagation paths of the independent fading effects. So it can be considered that the received signal averaged with respect to frequencies is equal to the signal averaged with respect to the propagation space. The space diversity is explained above to cause dB-normal distribution by the ergodic hypothesis. Then, the frequency diversity will become equal to the space diversity, which reduces Rayleigh fading to dB-normal distribution. By the above-mentioned discussion, Eqs. (5) and (6) lead to the fact that the space or frequency diversity reduces Rayleigh fading at its maximum by 1.8, 6.7, 13.0, 19.8 and 27.1 dB at cumulative values of 10, 1, 0.1, 0.01 and 0.001 %, respectively (the values of 10 % and 1 % are shown in Fig.2).

An example of fading reduction using a frequency-hopping multilevel FSK technique⁽²⁾ shows that the spreading spectrum gains the same effect as expected by increasing the signal level by about 9, 15, and 21 dB at bit error rates of 10^{-2} , 10^{-3} and 10^{-4} , respectively.

5. Conclusion: The reduction of Rayleigh fading is theoretically considered to know the upper limit of the reduction by space or frequency diversity, and the reduction is calculated using the relation between Rayleigh and dB-normal distributions. The calculated values agree with the experimental data within the range of errors caused on the assumption that the bit error rate is nearly proportional to the cumulative distribution rate.

References

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- (2) M. Mizuno, "Diversity Effect on FH-MFSK Land Mobile Radio Equipments," in this Summaries of Papers.