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THE ELECTRICAL OSCILLATIONS OF A PLANE ELLIPTIC CONDUCTOR

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Introduction

Since Möglich's paper of 1927[1], little substantial contribution to the problem of the electrical oscillations of a perfectly conducting ellipsoid has been made for a long time. This seems by the reason that the expressions of the ellipsoidal wave functions have not been given in a simple form to get the numerical values. But recently Arscott[2] and Müller[3] have investigated the solutions of the ellipsoidal wave equation and the characteristic values. In their papers, asymptotic representations of the wave function have been given in physically significant forms. So it may be said that the problem about the electrical oscillations of a conducting ellipsoid can be investigated in detail.

In this paper, the electrical free oscillations of a plane elliptic conductor whose shape is nearly equal to prolate or oblate spheroid are investigated using ellipsoidal wave functions. Also several radiation patterns are shown in figures.

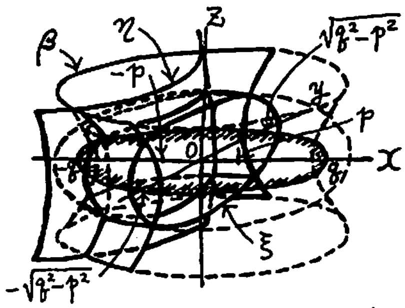

Fig. 1. Ellipsoidal coordinates

Analysis

In the ellipsoidal coordinate system, the solutions of the vector wave equation in a form directly applicable to the solution of boundary value problems are unfortunately not known. And so the problem of oscillations of ellipsoid in general form can not reduce to the boundary value problem to the scalor wave equation. But for the electrical oscillation problem of an elliptic plane conducting plate or of an elliptic hole on a conducting plane, the procedure discribed above can be applied successfully.

The solenoidal solutions of the vector wave equation

$$(\nabla^2+k^2)\mathbb{C}=0, \quad (k^2=\omega^2\varepsilon\mu, \text{ time factor}, \exp(j\omega t))$$

have the forms,

$$\mathbb{M}=\nabla\times\mathbb{a}\psi, \quad \mathbb{N}=k^{-1}\nabla\times\mathbb{M} \quad (\mathbb{a}=\mathbb{e} \text{ or } \mathbb{r})$$

where $(\nabla^2+k^2)\psi=0$, $\mathbb{e}$;constant unit vector, $\mathbb{r}$;position vector. The vector wave function($\mathbb{M}^e, \mathbb{N}^e$) and($\mathbb{M}^r, \mathbb{N}^r$) in a ellipsoidal coordinate system shown in Fig.1 have in the following forms

$$\mathbb{M}^x=q\Big[-\frac{\xi}{h_\eta h_\beta}\Big(dn\beta\frac{\partial\phi}{\partial\beta}-\eta\, dn'\beta\frac{\partial\phi}{\partial\eta}\Big)\mathbb{e}_\xi + \frac{\eta}{h_\beta h_\xi}\Big(dn\beta\frac{\partial\phi}{\partial\beta}-\xi\, dn'\beta\frac{\partial\phi}{\partial\xi}\Big)\mathbb{e}_\eta + \frac{dn\beta}{h_\xi h_\eta}\Big(\xi\frac{\partial\phi}{\partial\xi}-\eta\frac{\partial\phi}{\partial\eta}\Big)\mathbb{e}_\beta\Big]$$

$$\mathbb{M}^r=q^2\Big[\Big(\eta\frac{\partial\psi}{\partial\beta}+k^2 cn\beta\, cn'\beta\frac{\partial\psi}{\partial\eta}\Big)\frac{\mathbb{e}_\xi}{h_\eta h_\beta} + \Big(k'^2\xi\frac{\partial\psi}{\partial\beta}-k^2 cn\beta\, cn'\beta\frac{\partial\psi}{\partial\xi}\Big)\frac{\mathbb{e}_\eta}{h_\beta h_\xi} - \Big(k'^2\xi\frac{\partial\psi}{\partial\eta}+\eta\frac{\partial\psi}{\partial\xi}\Big)\frac{\mathbb{e}_\beta}{h_\xi h_\eta}\Big]$$

$$\mathbb{N}^{x,r}=k^{-1}\nabla\times\mathbb{M}^{x,r}$$

where $\mathbb{e}=\mathbb{e}_x$. The ellipsoidal coordinates(ξ, η, β) are defined in terms of the rectangular coordinates by the relations

$$x = g\xi\eta\, dn\beta,$$

$$y = g\sqrt{(1-\xi^2)(\eta^2-k'^2)}\, cn\beta$$

$$z = g\sqrt{(1-k'^2\xi^2)(\eta^2-1)}\, sn\beta$$

where $k' = p/g$, $k = \sqrt{g^2-p^2}/g$, $sn\beta$, $cn\beta$, $dn\beta$;Jacobian elliptic functions, h_ξ, h_η, h_β ;metrical coefficient, ϕ, ψ ;the solutions of $(\nabla^2+\kappa^2)V=0$. The boundary conditions for the scalor wave functions ϕ or ψ are $\phi=0$ or $\partial\psi/\partial\xi=0$ at $\xi=0$ for a plane elliptical plate conductor and $\partial\phi/\partial\eta=0$ or $\psi=0$ at $\eta=0$ for a conducting plate having an elliptical hole. The form of vector wave functions defined above coincides with the form for the prolate spheroidal coordinates(ξ, η, φ) in the limit of $k\to 0$ ($k'\to 1$). So this expression is useful for the problem of elliptic body which is nearly equal to thin wire or slit.

The ellipsoidal coordinates(ξ, η, γ) defined by the relations

$$x = \frac{g}{k'}\sqrt{(1+k^2\xi^2)(1-k^2\eta^2)}\, dn\gamma$$

$$y = g\frac{jk^2}{k'}\sqrt{(1+\xi^2)(1-\eta^2)}\, cn\gamma$$

$$z = -gk^2\xi\eta\, sn\gamma$$

are, on the other hand, useful for the problem of elliptic body which is nearly equal to circular disk or hole. In this case, the coordinates(ξ, η, γ) coinside with the oblate ones(ξ, η, φ)in the limit of $k\to 1$ ($k'\to 0$).

In the case of thin wire, the scalor wave equation $(\nabla^2+\kappa^2)V=0$ is separated to the following differential equations

$$(1-\xi^2)\{(1-\xi^2)-k^2(1-2\xi^2)\}\frac{d^2V_\xi}{d\xi^2}-\xi\{2(1-\xi^2)+k^2(3\xi^2-2)\}\frac{dV_\xi}{d\xi}+\{\omega^2\xi^4+E\xi^2+H-k^2\xi^2(2\omega^2\xi^2+E)\}V_\xi=0$$

$$(1-\eta^2)\{(1-\eta^2)-k^2\}\frac{d^2V_\eta}{d\eta^2}-2\eta\{(1-\eta^2)-\frac{k^2}{2}\}\frac{dV_\eta}{d\eta}+(\omega^2\eta^4+E\eta^2+H)V_\eta=0$$

$$\frac{d^2V_\beta}{d\beta^2}-\left[(\omega^2+E+H)-\frac{k^2}{2}(2\omega^2+E)+\frac{k^2}{2}(2\omega^2+E)\cos 2\beta\right]V_\beta=0$$

where $V_\xi V_\eta V_\beta = V$.

Those equations can be solved by using the perturbation procedures from the prolate spheroidal wave functions selecting k as a perturbation parameter. Especially the equation for V_β is Mathieu's differential equation. So the field pattern with respect to the argument β can be easily found. In Fig.2, the current and the electric intensity distributions in the equator plane are shown. Those results are directly applicable to the problems of elliptic plane conductor whose thickness is sufficiently thin compared to the wavelength.

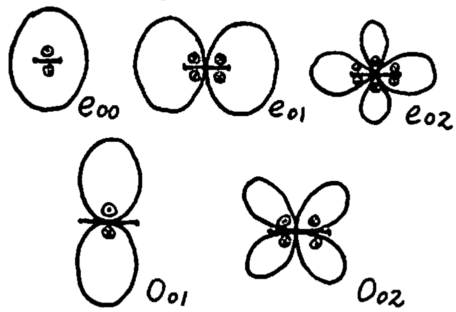

Fig.2. Field pattern.

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