SUFFICIENT CONDITION FOR A UNIQUE RECONSTRUCTION OF PENETRABLE OBJECT BY USING THE METHOD OF MOMENTS AND THE ITERATIVE ALGORITHM

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Abstract
The iterative method minimizing the cost function may be used for the reconstruction of a dielectric object from the measured scattered fields. It is shown that N+1 data values are sufficient to obtain a unique permittivity distribution of the object discretized into N small cells. Up to 3-cell object, equations derived from the formulation of the method of moments show this condition explicitly. For the object cells more than three, the simulated annealing algorithm with the Levenberg-Marquardt algorithm is used to obtain the global minimum of the cost function which shows numerically that N+1 or more data are needed to reconstruct the original distribution.

1. Introduction

The moment method inversion gives the reconstruction of the high-contrast dielectric object but its size is limited[1]. The method of moments requires the size of the discretized cell to be smaller than $0.2\lambda/\sqrt{\varepsilon_r}$[2], where λ is the free space wavelength and ε_r is the relative dielectric constant of the object. This moment method inversion is shown to suffer from the illposedness[3] in a sense that a small error in the scattered field causes a large error in the polarization current. This illposedness may be identified as the exponentially decaying behavior of the evanescent modes, which makes the small error in the scattered field grow exponentially in the back propagating process of the inversion[4]. Selecting only propagating modes excluding the evanescent(or exponentially small) modes of the scattered fields, the illposedness is stabilized[4] without the regularization[3] nor the pseudo-inversion[5].

For larger object to be reconstructed, an iterative algorithm minimizing the cost function is used, where the cost function is defined as the squared magnitude of the difference between the measured and calculated scattered fields from the assumed set of dielectric profile. For a low contrast object, an iterative Born method is used where the lossless dielectric object of $2\lambda \times 2\lambda$ is divided into 19×19 cells(i.e. 361 unknowns) and reconstructed by Wang and Chew[7] from the scattered fields of 36 points with 8 different incidences which makes the total number of 288 complex data or 576 real data. For a high-contrast large object, simulated annealing algorithm is used to find the global minimum of the cost function, Caorsi and Gragnani[7] used 128 data (16 field points $\times$ 4 incidences $\times$ 2) for the reconstruction of 16 cell object and Garnero and Pichot used 72 data (9 field points $\times$ 4 incidences $\times$ 2) for 25 cell object[8]. They use much larger data than the unknowns. Ra and Park[9] show that about the same number of data from 16 to 22 are used for reconstruction of 16 cell object by employing only the effective propagating modes in the spectral domain.

It is interesting to ask a question what is the sufficient number of data ensuring a unique reconstruction of the object by using the iterative method when the forward scattered field is calculated by the method of moments.

2. Sufficient condition for the lossless N-cell object reconstruction

Discretizing the dielectric object into N small cells and applying the method of moments[2], the scattered field u^s may be obtained as

$$u^s(\rho_m) = \sum_{n=1}^{N} \Delta\varepsilon_n u_n D_{mn} \quad , \tag{1}$$

where $\Delta\varepsilon_n = \varepsilon_n - 1$, ε_n and u_n are, respectively, the relative dielectric constant and the total field at the center of the nth cell, a is the radius of the equivalent circular cell, and

$$D_{mn} = -\frac{j\pi k_0 a}{2} J_1(k_0 a) H_0^{(2)}(k_0 \rho_{mn}). \tag{2}$$

Here J_1 is the first order Bessel function and the ρ_{mn} is the distance between the field point ρ_m and the center point of the nth cell ρ_n.

Substituting $u^s(\rho_n) = u(\rho_n) - u^i(\rho_n) = u_n - u_n^i$ into eq. (1), one obtains the total field u_n in terms of the incident fields u_n^i as

$$\sum_{l=1}^{N} C_{ln} u_n = u_n^i \; , \tag{3}$$

where

$$C_{ln} = \begin{cases} 1 + \frac{j}{2}[\pi k_0 a H_1^{(2)}(k_0 a) - 2j]\Delta\varepsilon_n & , \; l = n \; , \\ -D_{ln} & , \; l \neq n \; , \end{cases} \tag{4}$$

and $H_1^{(2)}$ is the 1st order Hankel function of the 2nd kind.

The cost function f may be defined as the summation of the squared magnitude of the difference between the measured scattering field(u_M^s) and the calculated scattering field(u_C^s) as

$$f = \frac{1}{2} \sum_{l=1}^{L} \sum_{i=1}^{I} \sum_{m=1}^{M} |F_{lim}|^2 \; , \quad F_{lim} = u_M^s(f_l; \theta_i; m) - u_C^s(f_l; \theta_i; m, \varepsilon_n^k) \; , \tag{5}$$

where f_l is the lth frequency, L is the total number of used frequencies, θ_i is the ith incident angle, I is the total number of incident waves, m is the mth measurement point, M is the total number of measurement points, and ε_n^k is the assumed dielectric profile of kth iteration.

One may investigate the sufficient conditions for the equations (1) and (3) or equivalently eq. (5) to have a unique solution of $\Delta\varepsilon_n$ by making $F_{lim}=0$, since u_M^s and u_C^s in eq. (5) corresponds to u^s and $\sum_{n=1}^{N} \Delta\varepsilon_n u_n D_{mn}$ in eq. (1), respectively, when the measured field u_M^s equals to the exact scattered field u^s in the ideal noiseless situation.

For one-cell object of lossless dielectric, i.e., N=1 and $\Delta\varepsilon_1$ is real, eq. (1) and (3) give $u_2^s = \Delta\varepsilon_1 u_1^i D_{21}/C_{11}$ or

$$\Delta\varepsilon_1\{a_1 \mathrm{Re}(u_2^s) - a_2 \mathrm{Im}(u_2^s) - a_3\} + j\Delta\varepsilon_1\{a_2 \mathrm{Re}(u_2^s) + a_1 \mathrm{Im}(u_2^s)\} = -\mathrm{Re}(u_2^s) + ja_4 - j\mathrm{Im}(u_2^s) \; . \tag{6}$$

which yields two real and imaginary equations, relating the complex scattered field u^s at one field point and $\Delta\varepsilon_1$, where the real coefficients, a_j, j=1, 2, 3, 4, are functions of k_0a and the distance between the object cell and the field point ρ_{21}. Real and imaginary parts of eq. (6) give two equations for $\Delta\varepsilon_1$ as a function of $\mathrm{Re}(u_2^s)$ and $\mathrm{Im}(u_2^s)$. Elimination of either one of $\mathrm{Re}(u_2^s)$ or $\mathrm{Im}(u_2^s)$ gives the 2nd order equation for $\Delta\varepsilon_1$ and both values of $\mathrm{Re}(u_2^s)$ and $\mathrm{Im}(u_2^s)$ are needed for a unique solution of $\Delta\varepsilon_1$.

For the lossless two-cell object,

$$b_1 \Delta\varepsilon_1 \Delta\varepsilon_2 + b_2 \Delta\varepsilon_1 + b_3 \Delta\varepsilon_2 + b_4 = 0 \tag{7}$$

may be obtained similarly from eq. (1) and (3), where the coefficients b_j are complex and given by $b_1 = \alpha_1 u_1^i + \alpha_2 u_2^i - \beta_1 u_3^s$, $b_2 = \alpha_3 u_1^i - \beta_2 u_3^s$, $b_3 = \alpha_4 u_1^i - \beta_2 u_3^s$, $b_4 = -u_3^s$, where α_j and β_j are functions of the geometry of the measurement and the cell distribution, u_1^i and u_2^i are incident waves at the cell 1 and 2, respectively, and u_3^s is the measured scattered fields at ρ_3. Real and imaginary parts of eq. (7), give a hyperbolic equation for $\Delta\varepsilon_1$ versus $\Delta\varepsilon_2$, respectively, and these two hyperbolic equations yield two sets of solutions. For a unique determination of $\Delta\varepsilon_1$ and $\Delta\varepsilon_2$, one more hyperbolic equation is needed and may be obtained from the measurement of the scattered field at another point.

For the lossless three-cell object, similar derivation yields

$$c_1\Delta\varepsilon_1\Delta\varepsilon_2\Delta\varepsilon_3 + c_2\Delta\varepsilon_1\Delta\varepsilon_2 + c_3\Delta\varepsilon_2\Delta\varepsilon_3 + c_4\Delta\varepsilon_2\Delta\varepsilon_3 + c_5\Delta\varepsilon_1 + c_6\Delta\varepsilon_2 + c_7\Delta\varepsilon_3 + c_8 = 0. \quad (8)$$

The real part of eq. (8) gives

$$\Delta\varepsilon_3 = -\frac{\mathrm{Re}(c_2)\Delta\varepsilon_1\Delta\varepsilon_2 + \mathrm{Re}(c_5)\Delta\varepsilon_1 + \mathrm{Re}(c_6)\Delta\varepsilon_2 + \mathrm{Re}(c_8)}{\mathrm{Re}(c_1)\Delta\varepsilon_1\Delta\varepsilon_2 + \mathrm{Re}(c_3)\Delta\varepsilon_1 + \mathrm{Re}(c_4)\Delta\varepsilon_2 + \mathrm{Re}(c_9)} \quad (9)$$

and substitution of eq. (9) into the imaginary part of eq. (8) gives

$$d_1\Delta\varepsilon_1^2\Delta\varepsilon_2^2 + d_2\Delta\varepsilon_1^2\Delta\varepsilon_2 + d_3\Delta\varepsilon_1\Delta\varepsilon_2^2 + d_4\Delta\varepsilon_1\Delta\varepsilon_2 + d_5\Delta\varepsilon_1^2 + d_6\Delta\varepsilon_2^2 + d_7\Delta\varepsilon_1 + d_8\Delta\varepsilon_2 + d_9 = 0 \ . \quad (10)$$

From the measurement of the scattered field at another point, two more equations like eq. (10) are generated from the real and the imaginary part of one complex equation. Numerical calculation, as shown in Fig. 1, for three cells of $0.06\lambda \times 0.06\lambda$ size located at $(-0.03\lambda, -0.03\lambda)$, $(0.03\lambda, -0.03\lambda)$, and $(-0.03\lambda, 0.03\lambda)$ and the scattered fields at two points, $(2\lambda, 0)$ and $(-2\lambda, 0)$, shows a unique solution at $\Delta\varepsilon_1$=6.0 and $\Delta\varepsilon_2$=6.0 crossing by 3 hyperbolic curves, where λ is the free space wavelength and the values of $\Delta\varepsilon_1$ and $\Delta\varepsilon_2$ are varied up to 100. $\Delta\varepsilon_3$ is then obtained from eq. (9). This shows that one needs two-point scattered fields, i.e., four real data for 3 unknowns, $\Delta\varepsilon_1$, $\Delta\varepsilon_2$, and $\Delta\varepsilon_3$.

For more cell problems than three, an iterative method using the hybrid algorithm combining the simulated annealing algorithm and the Levenberg-Marquardt algorithm[9] is used for finding the global minimum of the cost function defined in eq. (5). One may confirm the analytic solutions in equations from (6) to (8) by plotting the cost functions in Figs. 1, 2, and 3. Fig. 2 shows that two global minima for ε_1 are generated by one data, i.e., only the real part of u^s at one point. A unique value of 10 becomes the global minimum from 2 data, i.e., the real and the imaginary values of u^s at one point for one-cell problem. For the two-cell problem, multiple minima occur, as shown in the inverted logarithmic plot of Fig. 3(a), if one utilizes only the real part of one point scattered field. A unique minimum occurs as in Fig. 3(b) if 3 data , i.e., one complex field value and one real part of another field are used. Numerical calculations up to 16 cell problems are shown in table 1, all of which show that one needs N+1 data for a unique reconstruction of N-cell lossless object. The final value of the cost function are less than 10^{-31} and the root mean square error of the reconstructed permittivities are less than 10^{-11}, which shows the numerical stability and its convergence.

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Table 1 Number of data versus the number of unknowns

# of cells	# of scattered fields	# of scattered field data	Reconstruction RMSE	Minimum values of cost function
1	1	2	2.03e-16	7.70e-32
2	2	3	6.10e-16	1.41e-32
3	2	4	2.41e-16	2.55e-34
4	3	5	4.56e-16	9.00e-33
6	4	7	1.33e-14	1.25e-32
9	5	10	1.14e-12	4.72e-32
12	7	13	8.03e-11	7.19e-30
16	9	17	1.29e-11	6.59e-31

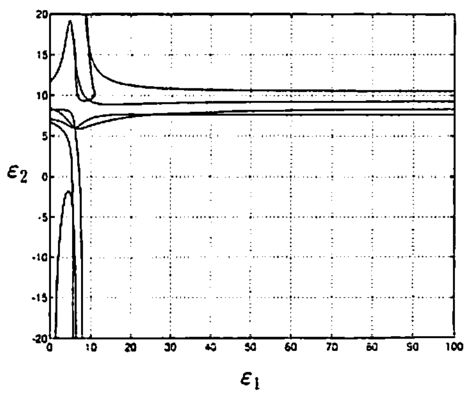

Fig. 1 Numerical solution of ε_1 and ε_2 from two point field values for three-cell reconstruction

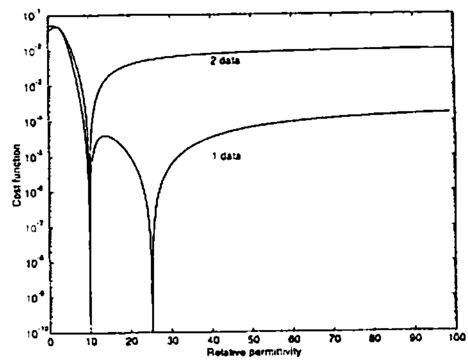

Fig. 2 Cost functions generated from $Re[u^s]$(1 data) and u^s (2 data)

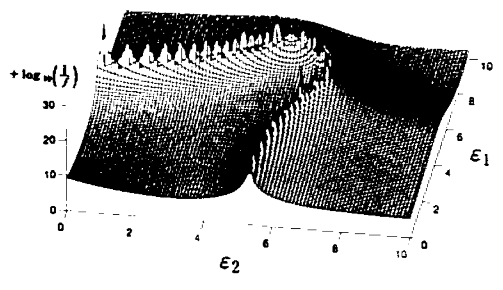

Fig. 3 (a) Mutiple minima of the cost function from 1 data of $Re[u^s]$ for 2-cell reconstruction

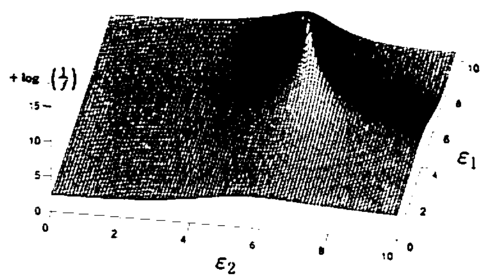

Fig. 3 (b) Single global minimum from 3 data for 2-cell reconstruction