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### A theorem on a solution curve of a class of nonlinear equations

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**Abstract**—This paper gives an important theorem on a solution curve of a class of nonlinear equations consisting of n variables and (n-1) equations, which is obtained from a typical nonlinear equation F(x) +Ax = b by deleting the first equation. The theorem is obtained under the assumption that A is an  $\Omega$ -matrix, which is a generalization of a P-matrix and a positive definite matrix. From this theorem and the previous results we can derive some very important properties on a solution curve. That is, the solution curves consist of only two types of curves.

### 1. Introduction

One of the authors once published the paper[1] which gave a necessary and sufficient conditions for a class of *n*-variable nonlinear equations F(x) + Ax = b to have a finite number of solutions. These equations are very important in nonlinear circuit theory and have been investigated for a long time in particular from a view point of the uniqueness of the solution.

In [1] Nishi and Kawane defined a new class of matrices called an  $\Omega$ -matrix, which is closely related to the maximum number of solutions. To investigate it, we have to investigate properties of solution curves of the equation obtained from F(x) + Ax = b by deleting the first equation. From the theorem we see that at the extremal point on a solution curve with respect to  $x_1$ , the inequality  $\frac{d^2 x_1}{dx_i^2} > 0$  holds for some *i*. This implies that solution curves have no maximal point with respect to  $x_1$ . As the results of the above we show that the solution curves consist of only two types of curves: One is a monotonically increasing curve with respect the variable  $x_1$  ranging from  $-\infty$  to  $+\infty$ , which means that the curve possesses neither local minimum nor global maximum with respect to  $x_1$ . The other has one global minimum (but not local minimum) with respect to  $x_1$ .

### 2. Formulation of the problem, an Ω-matrix and the previous results[1]

### 2.1. Formulation of the problem

We consider the nonlinear equations of n variables:

$$F(x) + Ax = b, (1)$$

where

$$F(x) = \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \\ \cdots \\ f_n(x_n) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

b is a constant vector, and  $f_i$   $(i = 1, 2, \dots, n)$  are sufficiently smooth (nonlinear) onto functions of  $x_i$  satisfying

$$\frac{d}{dx_i}f_i(x_i) > 0 \quad \text{for } \forall x_i \quad (i = 1, 2, \cdots, n) \quad (2)$$
$$\frac{d^2}{dx_i^2}f_i(x_i) > 0 \quad \text{for } \forall x_i \quad (i = 1, 2, \cdots, n) \quad (3)$$

### **2.2.** The $\Omega$ -matrix

In this paper the " $\Omega$ -matrix" defined below plays a definitely important role.

**Definition 1**: Let B be a real square matrix and assume that by using an appropriate permutation matrix P the matrix B can be changed into the form

$$PBP^{T} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \quad (B_{ii} \text{ are square matrices})$$
(4)

Then B is said to be a *reducible matrix*. The matrix which is not reducible is said to be *irreducible*.

Concerning the maximum number of solutions of Eq. (1) we can assume without loss of generality that A is irreducible.

**Definition 2:** Let *B* be a real square matrix of order *n*. We say that *B* satisfies the  $\Omega$ -sign condition if for each  $i(i = 1, 2, \dots, n)$ 

$$b_{ii} < 0$$
, then  $b_{ij} \le 0$  for  $\forall j \neq i$  (5)

**Definition 3** ( $\Omega$ -matrix): An *irreducible* matrix A is called an  $\Omega$ -matrix or is written as  $A \in \Omega$ , if  $(A+D)^{-1}$  satisfies the  $\Omega$ -sign condition for  $\forall D > 0$  satisfying  $|A + D| \neq 0.^1$ 

If A is reducible, then by an appropriate permutation matrix P it can be changed into a block upper triangle matrix of the form:

$$PAP^{T} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{kk} \end{bmatrix},$$
(6)

<sup>1</sup>Henceforth D,  $D_0$  and  $\hat{D}$  denote a positive diagonal matrix of an appropriate size.

where  $A_{ii}$   $(i = 1, \dots, k)$  are square and irreducible matrices. Then A is called an  $\Omega$ -matrix if all of  $A_{ii}$   $(i = 1, \dots, k)$  are  $\Omega$ -matrices.

The  $\Omega$ -matrix is closely related to the maximum number of solutions of Eq. (1).

### 3. Main theorems

#### 3.1. Solution curves

In this paper we will investigate about the last (n-1) equations of Eq. (1), i.e.,

$$\begin{bmatrix} f_2(x_2) \\ \cdots \\ f_n(x_n) \end{bmatrix} + \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_2 \\ \cdots \\ b_n \end{bmatrix}$$
(7)

We call Eq. (7) solution curve equations of Eq. (1) or simply SC equations. The set of points satisfying Eq. (7) is usually an assembly of one-dimensional curves in *n*-dimensional space. We call the set of those points solution curves or paths. Solution curves consist of several disconnected curves and each separated solution curve is denoted by C or  $C_i$  (see Fig. 1, for example).

The purpose of this paper is to clarify some properties of solution curves in order to investigate the maximum number of solutions of Eq. (1). Throughout this paper we assume

Assumption 1  $A \in \Omega$  and A is irreducible.

### **3.2.** Jacobian matrix $J_0$

The Jacobian matrix  $J_0$  of Eq. (7) is given as

$$J_{0} = \begin{bmatrix} a_{21} & a'_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a'_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a'_{nn} \end{bmatrix}$$
(8)

where

$$a'_{ii} = a_{ii} + d_i, \quad d_i = \frac{df_i}{dx_i} > 0$$
 (9)

In [2] we gave the following remarkable theorem: **Theorem 1**: The Jacobian matrix  $J_0$  in Eq.(8) is of full rank, i.e., rank $(J_0) = n - 1$  for  $\forall D > 0$  under Assumption 1.

From Theorem 1, implicit function theorem, and path theorem [3] we see that solution curves (paths) satisfying Eq. (7) are smooth and have neither cross point nor endpoint nor bifurcation point. Fig. 1 shows some typical examples of solution curves  $(C_1 \sim C_5)$ . Here the vertical axis denotes the variable  $x_1$  and the horizontal axis denotes the other (n - 1) variables  $x_2 \sim x_n$  denoted simply by  $X_0$ . So the figure is drawn to focus on the direction of  $x_1$ . The curve  $C_1$  is a continuous function ranging  $-\infty < x_1 < \infty$ ,  $C_2$  is a loop,  $C_3$  is a lower bounded curve,  $C_4$  is an upper bounded curve, and  $C_5$  is both upper and lower bounded with respect to  $x_1$ .



Fig. 1 Example of typical solution curves

## **3.3.** The maximal point of the solution curve with respect to $x_1$

Let C be a solution curve. An extremal point on C with respect to  $x_1$  means the point at which  $\frac{dx_1}{dp} = 0$  where p is the path parameter [3] or one of an appropriate variables among  $x_i$ .

The main purpose of this paper is to give the following important theorem:

**Theorem 2**: A solution curve has no maximal point with respect to  $x_1$  if  $A \in \Omega$  and if A is irreducible.

Proof is in Section 4.1.

As the result of this theorem we conclude that there is no points such as  $Q_1$ ,  $Q_2$ , and  $Q_3$  in Fig. 1.

**Corollary of Theorem 2**: The solution curves include no loop.

### **3.4.** Saturation of a solution curve with respect to $x_1$

Concerning saturation curve (see  $C_5$  in Fig. 1) in the direction of  $x_1$ , we have:

**Theorem 3**: Suppose that  $A \in \Omega$ . Then a solution curve does not saturate upward with respect to  $x_1$ , but can saturate downward.

Proof is in Section 4.2

#### **3.5.** Solution curves for $A \in \Omega$

By using Theorems 1, 2 and 3 we see that under Assumption 1 solution curves consist of only two types of curves:



Fig. 2 Example of typical solution curves for  $A \in \Omega$ 

One is a monotonically increasing curve with respect the variable  $x_1$  ranging from  $-\infty$  to  $+\infty$ (such as  $C_1$ ,  $C_2$  and  $C_4$  in Fig. 2(a)), which means that the curve neither possesses local and global maximum point nor saturates upward with respect to  $x_1$ . The other is a curve having the global minimum but not a local minimum (such as  $C_3$  and  $C_5$  in Fig. 2(a)). These characteristics are the same as those obtained in the case of n = 2.

### 4. Proof of Theorems 2 and 3

### 4.1. Notations of submatrices and minors of A

The notation  $A\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$   $(i_l < i_{l+1}; j_l < j_{l+1})$  denotes the matrix obtained from A by deleting the  $i_1$ -,  $i_2$ -,  $\cdots$ ,  $i_k$ -th rows and the  $j_1$ -,  $i_2$ -,  $\cdots$ ,  $j_k$ -th columns.

Similarly 
$$\Delta_A \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} = (-1)^{\sum_{h=1}^k i_h + \sum_{h=1}^k j_h} |A \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}|$$

### 4.2. Proof of Theorem 2

We want to prove Theorem 2 for general case of n. But for simplicity we describe the proof for the case of n = 5, because we can easily generalize it for n.

Let A' = A + D(D > 0). We consider the behavior of a solution curve C in the neighbor of an extremal point  $x_*$  with respect to  $x_1$ . So we can assume that

$$\Delta_{A'} \begin{pmatrix} 1\\1 \end{pmatrix} = 0 \quad \text{at } x_* \text{ on } C \tag{10}$$

Eq. (7) is rewritten as

$$f_i(x_i) + \sum_{j=1}^{5} a_{ij} x_j = b_i \ (i = 2, \cdots, 5)$$
 (11)

Since  $x_1$  is not appropriate as an independent variable around  $x_*$  on C, we use another variable among  $x_i(i = 2, \dots, 5)$  as an independent variable. For this purpose  $x_i$  has to be chosen so that  $\Delta_{A'} \begin{pmatrix} 1 \\ i \end{pmatrix} \neq 0$ . Since  $A \in \Omega$  (as well as  $A' \in \Omega$ ) and A is irreducible, we see from Theorem 2 that the rank of  $J_0$  is 4 and therefore under Eq. (10) there necessarily exists a nonzero cofactor among  $\Delta_{A'} \begin{pmatrix} 1 \\ i \end{pmatrix}$  (i = 2, 3, 4, 5). Without loss of generality we assume that

$$\Delta_{A'} \begin{pmatrix} 1\\5 \end{pmatrix} \begin{pmatrix} a_{21} & a'_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a'_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a'_{44}\\ a_{51} & a_{52} & a_{53} & a_{54} \end{pmatrix} \neq 0 \quad (12)$$

and we regard  $x_5$  as the independent variable for representing C.

Since

$$\Delta_{A'} \left( \begin{array}{c} 1\\5 \end{array} \right) = \sum_{i=2}^{5} a_{i1} \Delta_{A'} \left( \begin{array}{c} 1&i\\1&5 \end{array} \right)$$
(13)

we see that Lemma 1:

$$\Delta_{A'} \begin{pmatrix} 1 & i \\ 1 & 5 \end{pmatrix} \neq 0 \text{ for } \exists i \ (i = 2, 3, 4, 5) \tag{14}$$

We will investigate the behavior of  $x_1(x_5)$ ,  $x_2(x_5)$ ,  $x_3(x_5)$ , and  $x_4(x_5)$ . On the curve C Eq. (11) holds

identically. Differentiating Eq. (11) with respect to  $x_5$ , we have

$$\frac{df_i}{dx_i}\frac{dx_i}{dx_5} + \sum_{i=1}^4 a_{ij}\frac{dx_j}{dx_5} + a_{25} = 0 \ (i = 2, 3, 4)$$
(15)

$$\frac{df_5}{dx_5} + \sum_{j=1}^4 a_{5j} \frac{dx_j}{dx_5} + a_{55} = 0 \tag{16}$$

or in the matrix form

$$\begin{bmatrix} a_{21} & a'_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a'_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a'_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix} \begin{bmatrix} \frac{dx_1}{dx_5} \\ \frac{dx_2}{dx_5} \\ \frac{dx_3}{dx_5} \\ \frac{dx_4}{dx_5} \end{bmatrix} = \begin{bmatrix} -a_{25} \\ -a_{35} \\ -a_{45} \\ -a'_{55} \end{bmatrix}$$
(17)

from which we have

$$\frac{dx_i}{dx_5} = \frac{\Delta_{A'} \begin{pmatrix} 1\\i \end{pmatrix}}{\Delta_{A'} \begin{pmatrix} 1\\5 \end{pmatrix}} \quad (i = 2, 3, 4), \qquad (18)$$

Differentiating Eqs. (15)–(16) again with respect to  $x_5$ , we have

$$\frac{d^2 f_i}{dx_i^2} \left(\frac{dx_i}{dx_5}\right)^2 + \frac{df_i}{dx_i} \frac{d^2 x_i}{dx_5^2} + \sum_{j=1}^4 a_{2j} \frac{d^2 x_j}{dx_5^2} = 0 \quad (19)$$

$$\frac{d^2 f_5}{dx_5^2} + \sum_{j=1}^4 a_{5j} \frac{d^2 x_j}{dx_5^2} = 0 \tag{20}$$

or in the matrix form

$$\begin{bmatrix} a_{21} & a'_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a'_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a'_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix} \begin{bmatrix} \frac{d^2 x_1}{dx_2^2} \\ \frac{d^2 x_2}{dx_2^2} \\ \frac{d^2 x_3}{dx_2^2} \\ \frac{d^2 x_3}{dx_2^2} \end{bmatrix} = -\begin{bmatrix} \frac{d^2 f_2}{dx_2^2} \left(\frac{dx_2}{dx_2}\right)^2 \\ \frac{d^2 f_3}{dx_3^2} \left(\frac{dx_3}{dx_5}\right)^2 \\ \frac{d^2 f_4}{dx_4^2} \left(\frac{dx_4}{dx_5}\right)^2 \\ \frac{d^2 f_4}{dx_2^2} \\ \frac{d^2 f_5}{dx_2^2} \end{bmatrix}$$

$$(21)$$

We solve Eqs. (21) for  $\frac{d^2x_1}{dx_5^2}$  as

$$\frac{d^2 x_1}{dx_5^2} = \frac{1}{\Delta_{A'} \begin{pmatrix} 1\\5 \end{pmatrix}} \left[ \sum_{i=2}^4 \Delta_{A'} \begin{pmatrix} 1&i\\1&5 \end{pmatrix} \frac{d^2 f_i}{dx_i^2} \left( \frac{dx_i}{dx_5} \right)^2 + \Delta_{A'} \begin{pmatrix} 1&5\\1&5 \end{pmatrix} \frac{d^2 f_5}{dx_5^2} \right]$$
(22)

Our purpose is to show that  $\frac{d^2 x_1}{dx_5^2} > 0$ . Since  $\frac{d^2 f_i}{dx_i^2} \left(\frac{dx_i}{dx_5}\right)^2 \ge 0$ , we will prove the following sufficient conditions for  $\frac{d^2 x_1}{dx_5^2} > 0$ .

**Proposition 1**: If  $A \in \Omega$  and if A is irreducible, then

$$\frac{\Delta_{A'}^2 \begin{pmatrix} 1\\i \end{pmatrix} \Delta_{A'} \begin{pmatrix} 1&i\\1&5 \end{pmatrix}}{\Delta_{A'} \begin{pmatrix} 1\\5 \end{pmatrix}} \ge 0 \quad (i=2,3,4,5)$$
(23)

holds and at least one inequality holds in Eq. (23).

Proof of Theorem 2 is a little similar to that of Theorem 1 [2]. To prove the proposition we will prove Lemmas 2–4 below.

Let *E* be a diagonal matrix  $E = \text{diag} \quad [\epsilon_1, \epsilon_2, \cdots, \epsilon_n]$  where  $|\epsilon_i| (\neq 0)$  (i = 2, 3, 4, 5) are sufficiently small (more exactly  $0 < |\epsilon_i| < d_i$ ), while  $\epsilon_1$  satisfies  $-d_1 < \epsilon_1 < \infty$ . Then  $A' + E \in \Omega$ .

**Lemma 2**: Let  $\hat{B}$  be an  $m \times m \Omega$ -matrix and suppose that  $|\hat{B}'| = |\hat{B} + \hat{D}| = 0$   $(\hat{D} > 0)$ . Let  $\hat{E}$  be an  $m \times m$ diagonal matrix like E. Then elements of each row of the cofactor matrix of  $\hat{B}' + \hat{E}(\in \Omega)$  are all nonnegative or all nonpositive.

Proof) Note that  $|\hat{B}' + \hat{E}|$  is a multilinear polynomial of the variables  $\epsilon_i$  and does not vanish identically because of the existence of the term  $\prod_{i=1}^{m} \epsilon_i$ . Since  $|\hat{B}'| = 0$  holds,  $|\hat{B}' + \hat{E}|$  has no nonzero constant term and therefore can be both positive and negative depending on an appropriate choice of  $\epsilon_i$ . If a row of the cofactor matrix of  $(\hat{B}' + \hat{E}) \in \Omega$  includes both a positive element and a negative element, then we can choose  $\epsilon_i$  such that  $(\hat{B}' + \hat{E})^{-1}$  does not satisfy the  $\Omega$ -sign condition. This contradicts the assumption  $\hat{B}' + \hat{E} \in \Omega$  and completes the proof of Lemma 2.

Let  $\hat{B}' = A' \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and we apply Lemma 2 to  $\hat{B}' = A' \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

We can choose  $\epsilon_1(>0)$  sufficiently large so that

$$|A' + E| = \begin{vmatrix} a'_{11} + \epsilon_1 & a_1. \\ a_{.1} & (A' + E) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{vmatrix}$$
$$\approx \epsilon_1 \Delta_{A' + E} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
(24)

 $\Delta_{A'+E}\begin{pmatrix}i\\i\end{pmatrix} \approx \epsilon_1 \Delta_{A'+E}\begin{pmatrix}1&i\\1&i\end{pmatrix} \quad (i=2,3,4)(25)$ 

hold for almost all  $\epsilon_i$  ( $|\epsilon_i|$ : sufficiently small)( $i = 2, \dots, 5$ ). Lemma 3:

$$\Delta_{A'} \left(\begin{array}{c} 1\\i\end{array}\right) \neq 0 \quad (i=2,3,4) \tag{26}$$

**Proof of Lemma 3**) Using Eqs. (24) and (25). we have  $(A' + E)^{-1}$ 

Suppose for example  $\Delta_{A'} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$ . Then

$$\Delta_{A'+E} \begin{pmatrix} 1\\2 \end{pmatrix} \equiv 0 \text{ independent of the values } \epsilon_1 \quad (28)$$

The reason is the same as the proof of Theorem 1 (see Ref. [2]). Then we see that A' is reducible(see Ref. [2]). This contradicts with Assumption 1. This completes the proof of Lemma 3.

From Eq. (27) and Lemma 2 we have **Lemma 4**:

$$\frac{\Delta_{A'+E} \begin{pmatrix} 1 & i \\ 1 & 5 \end{pmatrix}}{\Delta_{A'+E} \begin{pmatrix} 1 \\ 5 \end{pmatrix}} \ge 0 \ (i=2,3,4,5) \tag{29}$$

From Lemmas 1, 3 and 4 we have Proposition 1.

### 4.3. Proof of Theorem 3

In this section we examine the behavior of the solution curve which is approaching to the saturation in the direction of  $x_1$ .

The proof of Theorem 3 can be obtained in a similar way as that of Theorem 2 by replacing Eqs. (10) with

$$\Delta_{A'} \begin{pmatrix} 1\\1 \end{pmatrix} \approx 0 \text{ at } x_* \tag{30}$$

### 5. Conclusions

This paper gives two important theorems on the solution curves of Eq. (7) under Assumption 1. As the results we see that there are only two types of solution curves.

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#### References

- T. Nishi and Y. Kawane, "On the number of solutions of nonlinear resistive circuits", IEICE Trans. on Fundamentals, vol. E74, no. 3, pp. 479–487, 1991.
- [2] T. Nishi, S. Oishi, and N. Takahashi, "On the rank of the Jacobian matrix of a class of nonlinear equations", IEICE Tech. Rept., CAS2011-127, pp.115–120, Mar. 2012(in Japanese).
- [3] W.I. Zangwill and C.B. Garcia, "Pathways to solutions, fixed points, and equilibria", Prentice-Hall. 1981.
- [4] M. Fiedler and V. Ptak, "On matrices with nonpositive off-diagonal elements and positive principal minors", Czech. Math. J., vol.12, pp.382-400. 1962.