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A theorem on a solution curve of a class of nonlinear equations

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Abstract—This paper gives an important theorem on a solution curve of a class of nonlinear equations consisting of n variables and $(n-1)$ equations, which is obtained from a typical nonlinear equation $F(x)+Ax=b$ by deleting the first equation. The theorem is obtained under the assumption that A is an Ω-matrix, which is a generalization of a P-matrix and a positive definite matrix. From this theorem and the previous results we can derive some very important properties on a solution curve. That is, the solution curves consist of only two types of curves.

1. Introduction

One of the authors once published the paper[1] which gave a necessary and sufficient conditions for a class of n-variable nonlinear equations $F(x)+Ax=b$ to have a finite number of solutions. These equations are very important in nonlinear circuit theory and have been investigated for a long time in particular from a view point of the uniqueness of the solution.

In [1] Nishi and Kawane defined a new class of matrices called an Ω-matrix, which is closely related to the maximum number of solutions. To investigate it, we have to investigate properties of solution curves of the equation obtained from $F(x)+Ax=b$ by deleting the first equation. From the theorem we see that at the extremal point on a solution curve with respect to x_1, the inequality $\frac{d^2x_1}{dx_i^2}>0$ holds for some i. This implies that solution curves have no maximal point with respect to x_1. As the results of the above we show that the solution curves consist of only two types of curves: One is a monotonically increasing curve with respect the variable x_1 ranging from $-\infty$ to $+\infty$, which means that the curve possesses neither local minimum nor global maximum *with respect to* x_1. The other has one global minimum (but not local minimum) with respect to x_1.

2. Formulation of the problem, an Ω-matrix and the previous results[1]

2.1. Formulation of the problem

We consider the nonlinear equations of n variables:

$$F(x)+Ax=b, \tag{1}$$

where

$$F(x)=\begin{bmatrix} f_1(x_1)\\ f_2(x_2)\\ \cdots\\ f_n(x_n)\end{bmatrix},\quad A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix},$$

b is a constant vector, and f_i $(i=1,2,\cdots,n)$ are sufficiently smooth (nonlinear) onto functions of x_i satisfying

$$\frac{d}{dx_i}f_i(x_i)>0 \quad \text{for } \forall x_i \quad (i=1,2,\cdots,n) \tag{2}$$

$$\frac{d^2}{dx_i^2}f_i(x_i)>0 \quad \text{for } \forall x_i \quad (i=1,2,\cdots,n) \tag{3}$$

2.2. The Ω-matrix

In this paper the "Ω-matrix" defined below plays a definitely important role.

Definition 1: Let B be a real square matrix and assume that by using an appropriate permutation matrix P the matrix B can be changed into the form

$$PBP^T=\begin{bmatrix} B_{11} & B_{12}\\ 0 & B_{22}\end{bmatrix} \quad (B_{ii} \text{ are square matrices}) \tag{4}$$

Then B is said to be a *reducible matrix*. The matrix which is not reducible is said to be *irreducible*.

Concerning the maximum number of solutions of Eq. (1) *we can assume without loss of generality that* A *is irreducible*.

Definition 2: Let B be a real square matrix of order n. We say that B satisfies the Ω*-sign condition* if for each $i(i=1,2,\cdots,n)$

$$b_{ii}<0, \text{ then } b_{ij}\le 0 \quad \text{for} \quad \forall j(\ne i) \tag{5}$$

Definition 3 (Ω-matrix): An *irreducible* matrix A is called an Ω*-matrix* or is written as $A\in\Omega$, if $(A+D)^{-1}$ satisfies the Ω-sign condition for $\forall D>0$ satisfying $|A+D|\ne 0$.[1]

If A is reducible, then by an appropriate permutation matrix P it can be changed into a block upper triangle matrix of the form:

$$PAP^T=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k}\\ 0 & A_{22} & \cdots & A_{2k}\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & A_{kk}\end{bmatrix}, \tag{6}$$

[1] Henceforth D, D_0 and $\hat{D}$ denote a positive diagonal matrix of an appropriate size.

where A_{ii} $(i = 1, \cdots, k)$ are square and irreducible matrices. Then A is called an *Ω-matrix* if all of A_{ii} $(i = 1, \cdots, k)$ are Ω-matrices.

The Ω-matrix is closely related to the maximum number of solutions of Eq. (1).

3. Main theorems

3.1. Solution curves

In this paper we will investigate about the last $(n-1)$ equations of Eq. (1), i.e.,

$$\begin{bmatrix} f_2(x_2) \\ \cdots \\ f_n(x_n) \end{bmatrix} + \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_2 \\ \cdots \\ b_n \end{bmatrix}. \quad (7)$$

We call Eq. (7) *solution curve equations* of Eq. (1) or simply *SC equations*. The set of points satisfying Eq. (7) is *usually* an assembly of one-dimensional curves in n-dimensional space. We call the set of those points *solution curves* or *paths*. Solution curves consist of several disconnected curves and each separated solution curve is denoted by C or C_i (see Fig. 1, for example).

The purpose of this paper is to clarify some properties of solution curves in order to investigate the maximum number of solutions of Eq. (1). Throughout this paper we assume

Assumption 1 $A \in \Omega$ and A is irreducible.

3.2. Jacobian matrix J_0

The Jacobian matrix J_0 of Eq. (7) is given as

$$J_0 = \begin{bmatrix} a_{21} & a'_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a'_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a'_{nn} \end{bmatrix} \quad (8)$$

where

$$a'_{ii} = a_{ii} + d_i, \quad d_i = \frac{df_i}{dx_i} > 0 \quad (9)$$

In [2] we gave the following remarkable theorem:

Theorem 1: The Jacobian matrix J_0 in Eq.(8) is of full rank, i.e., $\mathrm{rank}(J_0) = n-1$ for $\forall D > 0$ under Assumption 1.

From Theorem 1, *implicit function theorem*, and *path theorem* [3] we see that solution curves (paths) satisfying Eq. (7) are *smooth* and have *neither cross point nor endpoint nor bifurcation point.* Fig. 1 shows some typical examples of solution curves ($C_1 \sim C_5$). Here the vertical axis denotes the variable x_1 and the horizontal axis denotes the other $(n-1)$ variables $x_2 \sim x_n$ denoted simply by X_0. So the figure is drawn to focus on the direction of x_1. The curve C_1 is a continuous function ranging $-\infty < x_1 < \infty$, C_2 is a loop, C_3 is a lower bounded curve, C_4 is an upper bounded curve, and C_5 is both upper and lower bounded with respect to x_1.

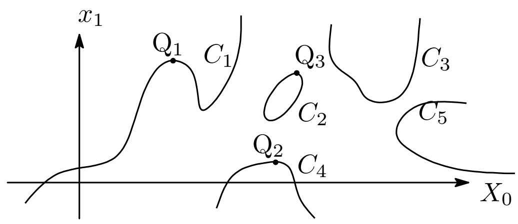

Fig. 1 Example of typical solution curves

3.3. The maximal point of the solution curve with respect to x_1

Let C be a solution curve. An extremal point on C with respect to x_1 means the point at which $\frac{dx_1}{dp} = 0$ where p is the path parameter [3] or one of an appropriate variables among x_i.

The main purpose of this paper is to give the following important theorem:

Theorem 2: A solution curve has no maximal point with respect to x_1 if $A \in \Omega$ and if A is irreducible.

Proof is in Section 4.1.

As the result of this theorem we conclude that there is no points such as Q_1, Q_2, and Q_3 in Fig. 1.

Corollary of Theorem 2: The solution curves include no loop.

3.4. Saturation of a solution curve with respect to x_1

Concerning saturation curve (see C_5 in Fig. 1) in the direction of x_1, we have:

Theorem 3: Suppose that $A \in \Omega$. Then a solution curve does not saturate upward with respect to x_1, but can saturate downward.

Proof is in Section 4.2

3.5. Solution curves for $A \in \Omega$

By using Theorems 1, 2 and 3 we see that under Assumption 1 solution curves consist of only two types of curves:

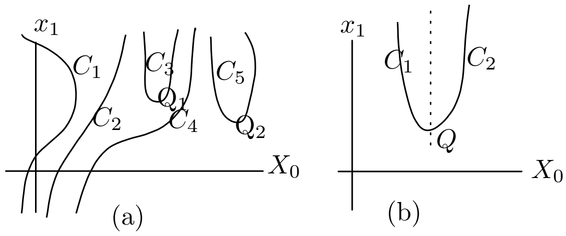

Fig. 2 Example of typical solution curves for $A \in \Omega$

One is a monotonically increasing curve with respect the variable x_1 ranging from $-\infty$ to $+\infty$(such as C_1, C_2 and C_4 in Fig. 2(a)), which means that the curve neither possesses local and global maximum point nor saturates upward *with respect to* x_1. The other is a curve having the global minimum but not a local minimum (such as C_3 and C_5 in Fig. 2(a)). These characteristics are the same as those obtained in the case of $n = 2$.

4. Proof of Theorems 2 and 3

4.1. Notations of submatrices and minors of A

The notation $A\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$ $(i_l < i_{l+1}; j_l < j_{l+1})$ denotes the matrix obtained from A by deleting the i_1-, i_2-, $\cdots$, i_k-th rows and the j_1-, j_2-, $\cdots$, j_k-th columns.

Similarly $\Delta_A\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} = (-1)^{\sum_{h=1}^k i_h + \sum_{h=1}^k j_h} \left| A\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} \right|$

4.2. Proof of Theorem 2

We want to prove Theorem 2 for general case of n. But for simplicity we describe the proof for the case of $n = 5$, because we can easily generalize it for n.

Let $A' = A + D (D > 0)$. We consider the behavior of a solution curve C in the neighbor of an extremal point x_* with respect to x_1. So we can assume that

$$\Delta_{A'}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \quad \text{at } x_* \text{ on } C \tag{10}$$

Eq. (7) is rewritten as

$$f_i(x_i) + \sum_{j=1}^{5} a_{ij} x_j = b_i \ (i = 2, \cdots, 5) \tag{11}$$

Since x_1 is not appropriate as an independent variable around x_* on C, we use another variable among $x_i (i = 2, \cdots, 5)$ as an independent variable. For this purpose x_i has to be chosen so that $\Delta_{A'}\begin{pmatrix} 1 \\ i \end{pmatrix} \neq 0$. Since $A \in \Omega$ (as well as $A' \in \Omega$) and A is irreducible, we see from Theorem 2 that the rank of J_0 is 4 and therefore under Eq. (10) there necessarily exists a nonzero cofactor among $\Delta_{A'}\begin{pmatrix} 1 \\ i \end{pmatrix}$ $(i = 2, 3, 4, 5)$. Without loss of generality we assume that

$$\Delta_{A'}\begin{pmatrix} 1 \\ 5 \end{pmatrix} \left(= \begin{vmatrix} a_{21} & a'_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a'_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a'_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{vmatrix} \right) \neq 0 \tag{12}$$

and we regard x_5 as the independent variable for representing C.

Since

$$\Delta_{A'}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \sum_{i=2}^{5} a_{i1} \Delta_{A'}\begin{pmatrix} 1 & i \\ 1 & 5 \end{pmatrix} \tag{13}$$

we see that

Lemma 1:

$$\Delta_{A'}\begin{pmatrix} 1 & i \\ 1 & 5 \end{pmatrix} \neq 0 \ \text{ for } \ \exists i \ (i = 2, 3, 4, 5) \tag{14}$$

We will investigate the behavior of $x_1(x_5)$, $x_2(x_5)$, $x_3(x_5)$, and $x_4(x_5)$. On the curve C Eq. (11) holds identically. Differentiating Eq. (11) with respect to x_5, we have

$$\frac{df_i}{dx_i}\frac{dx_i}{dx_5} + \sum_{j=1}^{4} a_{ij}\frac{dx_j}{dx_5} + a_{25} = 0 \ (i = 2, 3, 4) \tag{15}$$

$$\frac{df_5}{dx_5} + \sum_{j=1}^{4} a_{5j}\frac{dx_j}{dx_5} + a_{55} = 0 \tag{16}$$

or in the matrix form

$$\begin{bmatrix} a_{21} & a'_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a'_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a'_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix} \begin{bmatrix} \frac{dx_1}{dx_5} \\ \frac{dx_2}{dx_5} \\ \frac{dx_3}{dx_5} \\ \frac{dx_4}{dx_5} \end{bmatrix} = \begin{bmatrix} -a_{25} \\ -a_{35} \\ -a_{45} \\ -a'_{55} \end{bmatrix} \tag{17}$$

from which we have

$$\frac{dx_i}{dx_5} = \frac{\Delta_{A'}\begin{pmatrix} 1 \\ i \end{pmatrix}}{\Delta_{A'}\begin{pmatrix} 1 \\ 5 \end{pmatrix}} \quad (i = 2, 3, 4), \tag{18}$$

Differentiating Eqs. (15)–(16) again with respect to x_5, we have

$$\frac{d^2 f_i}{dx_i^2}\left(\frac{dx_i}{dx_5}\right)^2 + \frac{df_i}{dx_i}\frac{d^2 x_i}{dx_5^2} + \sum_{j=1}^{4} a_{2j}\frac{d^2 x_j}{dx_5^2} = 0 \tag{19}$$

$$\frac{d^2 f_5}{dx_5^2} + \sum_{j=1}^{4} a_{5j}\frac{d^2 x_j}{dx_5^2} = 0 \tag{20}$$

or in the matrix form

$$\begin{bmatrix} a_{21} & a'_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a'_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a'_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix} \begin{bmatrix} \frac{d^2x_1}{dx_5^2} \\ \frac{d^2x_2}{dx_5^2} \\ \frac{d^2x_3}{dx_5^2} \\ \frac{d^2x_4}{dx_5^2} \end{bmatrix} = - \begin{bmatrix} \frac{d^2 f_2}{dx_2^2}\left(\frac{dx_2}{dx_5}\right)^2 \\ \frac{d^2 f_3}{dx_3^2}\left(\frac{dx_3}{dx_5}\right)^2 \\ \frac{d^2 f_4}{dx_4^2}\left(\frac{dx_4}{dx_5}\right)^2 \\ \frac{d^2 f_5}{dx_5^2} \end{bmatrix} \tag{21}$$

We solve Eqs. (21) for $\frac{d^2x_1}{dx_5^2}$ as

$$\frac{d^2 x_1}{dx_5^2} = \frac{1}{\Delta_{A'}\begin{pmatrix} 1 \\ 5 \end{pmatrix}} \left[\sum_{i=2}^{4} \Delta_{A'}\begin{pmatrix} 1 & i \\ 1 & 5 \end{pmatrix} \frac{d^2 f_i}{dx_i^2}\left(\frac{dx_i}{dx_5}\right)^2 + \Delta_{A'}\begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix} \frac{d^2 f_5}{dx_5^2} \right] \tag{22}$$

Our purpose is to show that $\frac{d^2x_1}{dx_5^2} > 0$. Since $\frac{d^2 f_i}{dx_i^2}\left(\frac{dx_i}{dx_5}\right)^2 \geq 0$, we will prove the following sufficient conditions for $\frac{d^2x_1}{dx_5^2} > 0$.

Proposition 1: If $A \in \Omega$ and if A is irreducible, then

$$\frac{\Delta_{A'}^2\begin{pmatrix} 1 \\ i \end{pmatrix} \Delta_{A'}\begin{pmatrix} 1 & i \\ 1 & 5 \end{pmatrix}}{\Delta_{A'}\begin{pmatrix} 1 \\ 5 \end{pmatrix}} \geq 0 \ \ (i = 2, 3, 4, 5) \tag{23}$$

holds and at least one inequality holds in Eq. (23).

Proof of Theorem 2 is a little similar to that of Theorem 1 [2]. To prove the proposition we will prove Lemmas 2–4 below.

Let E be a diagonal matrix $E = \mathrm{diag}\ [\epsilon_1, \epsilon_2, \cdots, \epsilon_n]$ where $|\epsilon_i|(\neq 0)$ $(i = 2,3,4,5)$ are sufficiently small (more exactly $0 < |\epsilon_i| < d_i$), while ϵ_1 satisfies $-d_1 < \epsilon_1 < \infty$. Then $A' + E \in \Omega$.

Lemma 2: Let $\hat{B}$ be an $m \times m$ Ω-matrix and suppose that $|\hat{B}'| = |\hat{B} + \hat{D}| = 0$ $(\hat{D} > 0)$. Let $\hat{E}$ be an $m \times m$ diagonal matrix like E. Then elements of each row of the cofactor matrix of $\hat{B}' + \hat{E}(\in \Omega)$ are all nonnegative or all nonpositive.

Proof) Note that $|\hat{B}' + \hat{E}|$ is a multilinear polynomial of the variables ϵ_i and does not vanish identically because of the existence of the term $\prod_{i=1}^{m} \epsilon_i$. Since $|\hat{B}'| = 0$ holds, $|\hat{B}' + \hat{E}|$ has no nonzero constant term and therefore can be both positive and negative depending on an appropriate choice of ϵ_i. If a row of the cofactor matrix of $(\hat{B}' + \hat{E})(\in \Omega)$ includes both a positive element and a negative element, then we can choose ϵ_i such that $(\hat{B}' + \hat{E})^{-1}$ does not satisfy the Ω-sign condition. This contradicts the assumption $\hat{B}' + \hat{E} \in \Omega$ and completes the proof of Lemma 2.

Let $\hat{B}' = A'\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and we apply Lemma 2 to $\hat{B}' = A'\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We can choose $\epsilon_1(> 0)$ sufficiently large so that

$$|A' + E| = \begin{vmatrix} a'_{11} + \epsilon_1 & a_{1\cdot} \\ a_{\cdot 1} & (A' + E)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{vmatrix} \approx \epsilon_1 \Delta_{A'+E}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (24)$$

$$\Delta_{A'+E}\begin{pmatrix} i \\ i \end{pmatrix} \approx \epsilon_1 \Delta_{A'+E}\begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix} \quad (i = 2,3,4) \quad (25)$$

hold for almost all ϵ_i ($|\epsilon_i|$: sufficiently small)$(i = 2, \cdots, 5)$.

Lemma 3:

$$\Delta_{A'}\begin{pmatrix} 1 \\ i \end{pmatrix} \neq 0 \quad (i = 2,3,4) \quad (26)$$

Proof of Lemma 3) Using Eqs. (24) and (25). we have

$$(A' + E)^{-1} \approx \frac{1}{\epsilon_1 \Delta_{A'+E}\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \begin{bmatrix} \Delta_{A'+E}\begin{pmatrix} 1 \\ 1 \end{pmatrix} & \Delta_{A'+E}\begin{pmatrix} 2 \\ 1 \end{pmatrix} & \cdots & \Delta_{A'+E}\begin{pmatrix} 5 \\ 1 \end{pmatrix} \\ \Delta_{A'+E}\begin{pmatrix} 1 \\ 2 \end{pmatrix} & \epsilon_1 \Delta_{A'+E}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} & \cdots & \epsilon_1 \Delta_{A'+E}\begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta_{A'+E}\begin{pmatrix} 1 \\ 5 \end{pmatrix} & \epsilon_1 \Delta_{A'+E}\begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix} & \cdots & \epsilon_1 \Delta_{A'+E}\begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix} \end{bmatrix} \quad (27)$$

Suppose for example $\Delta_{A'}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$. Then

$$\Delta_{A'+E}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \equiv 0 \text{ independent of the values } \epsilon_1 \quad (28)$$

The reason is the same as the proof of Theorem 1 (see Ref. [2]). Then we see that A' is reducible(see Ref. [2]). This contradicts with Assumption 1. This completes the proof of Lemma 3.

From Eq. (27) and Lemma 2 we have

Lemma 4:

$$\frac{\Delta_{A'+E}\begin{pmatrix} 1 & i \\ 1 & 5 \end{pmatrix}}{\Delta_{A'+E}\begin{pmatrix} 1 \\ 5 \end{pmatrix}} \geq 0 \ (i = 2,3,4,5) \quad (29)$$

From Lemmas 1, 3 and 4 we have Proposition 1.

4.3. Proof of Theorem 3

In this section we examine the behavior of the solution curve which is approaching to the saturation in the direction of x_1.

The proof of Theorem 3 can be obtained in a similar way as that of Theorem 2 by replacing Eqs. (10) with

$$\Delta_{A'}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \approx 0 \text{ at } x_* \quad (30)$$

5. Conclusions

This paper gives two important theorems on the solution curves of Eq. (7) under Assumption 1. As the results we see that there are only two types of solution curves.

Acknowledgments

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