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Semi-discrete Analogues of the Elastic Beam Equation and the Short Pulse Equation

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Abstract—Two integrable nonlinear differential-difference systems, semi-discrete analogues of the Wadati-Konno-Ichikawa elastic beam equation and the short pulse equation, are constructed by using a geometric approach.

1. Introduction

In 1975, Ablowitz and Ladik proposed a method of obtaining certain classes of integrable nonlinear differential-difference equations which are semi-discrete analogues of integrable nonlinear partial differential equations such as the nonlinear Schrödinger (NLS) equation

$$i\psi_t = \psi_{xx} \pm 2|\psi|^2\psi\,, \tag{1}$$

and the modified KdV (mKdV) equation

$$q_t = q_{xxx} \pm 6q^2 q_x\,, \tag{2}$$

based on the Ablowitz-Kaup-Newell-Segur (AKNS) spectral problem [1,2]. As a discrete analogue of the NLS equation, Ablowitz and Ladik obtained

$$i\frac{d\psi_n}{dt} = \frac{1}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) \pm |\psi_n|^2(\psi_{n+1} + \psi_{n-1})\,, \tag{3}$$

which is often called the Ablowitz-Ladik equation. They also obtained a discrete analogue of the mKdV equation

$$\frac{dq_n}{dt} = (1 \pm h^2 q_n^2)(q_{n+1} - q_{n-1})\,. \tag{4}$$

These nonlinear differential-difference equations have exact N-soliton solutions.

It is known that there is a class of integrable nonlinear partial differential equations which admit singularities such as loop, cusp, and peak soliton solutions. Among them, the Wadati-Konno-Ichikawa (WKI) elastic beam equation

$$u_t = -\left(\frac{u_x}{(1+u^2)^{\frac{3}{2}}}\right)_{xx}\,, \tag{5}$$

and the short pulse equation

$$v_{xt} = 4v + \frac{2}{3}\left(v^3\right)_{xx}\,, \tag{6}$$

appear in various physical phenomena [3–7]. For example, the short pulse equation (6) describes ultra short optical pulses which cannot be described by the NLS equation (1), thus we can consider the short pulse equation as an extension of the NLS equation [7]. It should be noted that the WKI elastic beam equation and the short pulse equation are transformed to the potential mKdV equation and the sine-Gordon equation, respectively, through hodograph (reciprocal) transformations [8–10].

In this contribution, we construct semi-discrete analogues of the WKI elastic beam equation and the short pulse equation by using a geometric approach. There have been intensive studies in topics related to curve geometry after the pioneering work of Lamb and Goldstein-Petrich [11, 12], and then several frameworks for the motion of discrete curves have been proposed in various settings [13, 14]. It is well known that the potential mKdV equation describes the motion of plane curves [12]. In this point of view, the hodograph (reciprocal) transformation of the WKI elastic beam equation and the short pulse equation can be viewed as the Euler-Lagrange transformation of the motion of plane curves. From this fact, we can discretize the WKI elastic beam equation and the short pulse equation by considering a discrete analogue of the hodograph (reciprocal) transformation for the motion of discrete curves.

2. Motion of smooth curves, the WKI elastic beam equation and the short pulse equation

Let $\gamma(s)$ be an arc-length parametrized curve in Euclidean plane $\mathbb{R}^2$. Then the tangent vector $\frac{\partial\gamma}{\partial s}$ (= $\boldsymbol{T}$) satisfies

$$\left|\frac{\partial\gamma}{\partial s}\right| = 1. \tag{7}$$

Thus $\frac{\partial\gamma}{\partial s}$ admits the parametrization

$$\boldsymbol{T} = \frac{\partial\gamma}{\partial s} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}. \tag{8}$$

The function $\theta = \theta(s)$ is called the angle function of γ which denotes the angle of $\frac{\partial\gamma}{\partial s}$ measured from the x-axis.

We define the normal vector $\boldsymbol{N}$ by

$$\boldsymbol{N} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial \gamma}{\partial s} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \tag{9}$$

and introduce the Frenet frame

$$F = (\boldsymbol{T}, \boldsymbol{N}), \tag{10}$$

which is the orthonormal basis attached to the curve. The Frenet equation is given by

$$\frac{\partial}{\partial s} F = F \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}, \tag{11}$$

where the function $\kappa = \frac{\partial \theta}{\partial s}$ is the curvature of γ. The angle function θ is also referred to as the potential function. Let us consider the following isoperimetric motion in time t:

$$\frac{\partial}{\partial t} F = F \begin{bmatrix} 0 & \kappa_{ss} + \frac{\kappa^3}{2} \\ -\kappa_{ss} - \frac{\kappa^3}{2} & 0 \end{bmatrix}. \tag{12}$$

In terms of $\frac{\partial \gamma}{\partial s}$, (11) and (12) can be expressed as

$$\frac{\partial}{\partial s}\left(\frac{\partial \gamma}{\partial s}\right) = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix} \frac{\partial \gamma}{\partial s}, \tag{13}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \gamma}{\partial s}\right) = \begin{bmatrix} 0 & \kappa_{ss} + \frac{\kappa^3}{2} \\ -\kappa_{ss} - \frac{\kappa^3}{2} & 0 \end{bmatrix} \frac{\partial \gamma}{\partial s}, \tag{14}$$

respectively. Then the compatibility condition of (11) and (12), or (13) and (14) yields the mKdV equation for $\kappa = \kappa(s,t)$ [11,12]

$$\kappa_t + \frac{3}{2}\kappa^2 \kappa_s + \kappa_{sss} = 0, \tag{15}$$

or the potential mKdV equation for $\theta = \theta(s,t)$:

$$\theta_t + \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0. \tag{16}$$

The mKdV equation can be viewed as the governing equation of the Lagrangian description for the motion of the curves γ in terms of the arc-length parameter s. Let us consider the Eulerian description of the same motion of the curves. We introduce the Eulerian coordinates

$$\gamma(s,t) = \begin{bmatrix} x(s,t) \\ v(s,t) \end{bmatrix} = \int_0^s \begin{bmatrix} \cos\theta(s',t) \\ \sin\theta(s',t) \end{bmatrix} ds' + \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}, \tag{17}$$

and change the independent variables (s,t) to

$$(x, t') = \left(\int_0^s \cos\theta(s',t)\, ds' + x_0,\ t \right). \tag{18}$$

For simplicity we write t' as t without causing confusion. Let us write down the equation for v in terms of x and t. It can be easily shown that

$$s(x,t) = \int \sqrt{1+v_x^2}\, dx,\ \kappa(x,t) = \frac{v_{xx}}{(1+v_x^2)^{\frac{3}{2}}}, \tag{19}$$

$$\boldsymbol{N} = \frac{1}{\sqrt{1+v_x^2}} \begin{bmatrix} -v_x \\ 1 \end{bmatrix}, \boldsymbol{T} = \frac{1}{\sqrt{1+v_x^2}} \begin{bmatrix} 1 \\ v_x \end{bmatrix}. \tag{20}$$

By using

$$\frac{\partial}{\partial t}\gamma = -\kappa_s \boldsymbol{N} - \frac{1}{2}\kappa^2 \boldsymbol{T}, \tag{21}$$

and relations $\boldsymbol{T} \cdot \boldsymbol{N} = 0$, $\boldsymbol{N} \cdot \boldsymbol{N} = 1$, it follows that

$$-\kappa_s = \gamma_t \cdot \boldsymbol{N} = \frac{v_t}{\sqrt{1+v_x^2}}, \tag{22}$$

by taking the inner product with $\boldsymbol{N}$ on both sides of (21). By using $\frac{ds}{dx} = \sqrt{1+v_x^2}$, we see that

$$v_t = -\kappa_s \sqrt{1+v_x^2} = -\kappa_x. \tag{23}$$

Thus we derive

$$v_t = -\left(\frac{v_{xx}}{(1+v_x^2)^{\frac{3}{2}}} \right)_x. \tag{24}$$

Introducing $u = v_x$, we obtain the WKI elastic beam equation (5) [3–6] Therefore, (24) or (5) can be viewed as the governing equation of the Eulerian description for the curve motions given by (11) and (12).

The above discussion shows that the hodograph (reciprocal) transformation arises naturally as the transformation between the Lagrangian and Eulerian descriptions from the point of view of geometry of plane curves.

It is well known that the sine-Gordon equation

$$\theta_{ts} = 4\sin\theta, \tag{25}$$

belongs to the mKdV hierarchy [15,16] and it describes a certain motion of plane curves [17]. It is possible to derive the governing equation of curve motion in the Eulerian description in a similar manner to the case of the mKdV equation. Applying the transformations

$$(x, t') = \left(\int_0^s \cos\theta(s',t)\, ds' + x_0,\ t \right), \tag{26}$$

$$v = \int_0^s \sin\theta(s',t)\, ds' + v_0, \tag{27}$$

we obtain the short pulse equation (6) where we set $t' = t$ for simplicity [7]. Again, we note that the short pulse equation (6) describes the same curve motions as the sine-Gordon equation by using the Eulerian description. The transformation (26) gives the hodograph (reciprocal) transformation between them [9,18,19].

3. Continuous motion of discrete curves and semi-discrete analogues of the WKI elastic beam equation and the short pulse equation

3.1. Continuous motion of a discrete curve and a semi-discrete WKI elastic beam equation

A map $\gamma : \mathbb{Z} \to \mathbb{R}^2;\ n \mapsto \gamma_n$ is said to be a discrete curve of segment length a_n if

$$\left| \frac{\gamma_{n+1} - \gamma_n}{a_n} \right| = 1. \tag{28}$$

We introduce the angle function ψ_n of a discrete curve γ by

$$\frac{\gamma_{n+1}-\gamma_n}{a_n} = \begin{bmatrix} \cos\psi_n \\ \sin\psi_n \end{bmatrix}. \tag{29}$$

A discrete curve γ satisfies

$$\frac{\gamma_{n+1}-\gamma_n}{a_n} = R(\kappa_n)\,\frac{\gamma_n-\gamma_{n-1}}{a_{n-1}}, \tag{30}$$

for $\kappa_n = \psi_n - \psi_{n-1}$, where $R(\kappa_n)$ denotes the rotation matrix given by

$$R(\kappa_n) = \begin{pmatrix} \cos\kappa_n & -\sin\kappa_n \\ \sin\kappa_n & \cos\kappa_n \end{pmatrix}. \tag{31}$$

We set $a_n = \epsilon\,(> 0)$, and consider the following motion of discrete curves:

$$\frac{d\gamma_n}{dt} = \frac{1}{\cos\frac{\kappa_n}{2}}\,R\left(-\frac{\kappa_n}{2}\right)\frac{\gamma_{n+1}-\gamma_n}{\epsilon}. \tag{32}$$

Then from the isoperimetric condition (28) and the compatibility condition of (30) and (32) , it follows that there exists a potential function θ_n characterized by

$$\psi_n = \frac{\theta_{n+1}+\theta_n}{2}, \quad \kappa_n = \frac{\theta_{n+1}-\theta_{n-1}}{2}, \tag{33}$$

and that θ_n satisfies the semi-discrete potential mKdV equation [1, 13, 20]

$$\frac{d\theta_n}{dt} = \frac{2}{\epsilon}\tan\left(\frac{\theta_{n+1}-\theta_{n-1}}{4}\right). \tag{34}$$

We note that $K_n = \frac{2}{\epsilon}\tan\frac{\kappa_n}{2}$ satisfies the semi-discrete mKdV equation

$$\frac{dK_n}{dt} = \frac{2}{\epsilon}\left(1+\frac{\epsilon^2}{4}K_n^2\right)(K_{n+1}-K_{n-1}). \tag{35}$$

It is possible to consider the Eulerian description of the curve motion defined by (30) and (32). Noting (29) and (33), we introduce the Eulerian coordinates

$$\gamma_n(t) = \begin{bmatrix} X_n(t) \\ v_n(t) \end{bmatrix} = \sum_{j=0}^{n-1}\begin{bmatrix} \epsilon\cos\left(\frac{\theta_{j+1}+\theta_j}{2}\right) \\ \epsilon\sin\left(\frac{\theta_{j+1}+\theta_j}{2}\right) \end{bmatrix} + \begin{bmatrix} X_0 \\ v_0 \end{bmatrix}. \tag{36}$$

Thus the angle function $\psi_n = \frac{\theta_{n+1}+\theta_n}{2}$ satisfies

$$\cos\psi_n = \frac{X_{n+1}-X_n}{\epsilon}, \quad \sin\psi_n = \frac{v_{n+1}-v_n}{\epsilon}, \qquad \tan\psi_n = \frac{v_{n+1}-v_n}{X_{n+1}-X_n}, \tag{37}$$

which can be regarded as the hodograph (reciprocal) transformation for (34).

Then from (29), (34) and (36), we derive

$$\frac{d}{dt}(X_{n+1}-X_n) = -\frac{v_{n+1}-v_n}{\epsilon}\,(G_{n+1}+G_n), \tag{38}$$

$$\frac{d}{dt}(v_{n+1}-v_n) = \frac{\delta_n}{\epsilon}\,(G_{n+1}+G_n), \tag{39}$$

where

$$G_n = \frac{v_{n+1}-2v_n+v_{n-1}}{X_{n+1}-2X_n+X_{n-1}}. \tag{40}$$

Note that v_n and X_n satisfy

$$\left(\frac{v_{n+1}-v_n}{\epsilon}\right)^2 + \left(\frac{X_{n+1}-X_n}{\epsilon}\right)^2 = 1. \tag{41}$$

From (38) and (39), we obtain

$$\frac{d}{dt}\left(\frac{v_{n+1}-v_n}{X_{n+1}-X_n}\right) = \frac{1}{\epsilon}\left(1+\left(\frac{v_{n+1}-v_n}{X_{n+1}-X_n}\right)^2\right)(G_{n+1}+G_n)\,. \tag{42}$$

The system of (38), (39) and (40) is nothing but the semi-discrete WKI elastic beam equation. Note that a different form was obtained in [22], but the above form is simpler than it.

Thus (42) can be rewritten as

$$\frac{d}{dt}\psi_n = \frac{1}{\epsilon}\left(\tan\frac{\psi_{n+1}-\psi_n}{2} + \tan\frac{\psi_n-\psi_{n-1}}{2}\right). \tag{43}$$

Equation (43) with the discrete hodograph (reciprocal) transformation

$$X_n(t) = \sum_{j=0}^{n-1}\epsilon\cos\psi_j(t) + X_0\,, \quad v_n(t) = \sum_{j=0}^{n-1}\epsilon\sin\psi_n(t) + v_0\,, \tag{44}$$

can be also regarded as the semi-discrete WKI elastic beam equation. In the continuous limit $\epsilon \to 0$ with $s = \epsilon n + t$ and $t' = -\frac{\epsilon^2}{6}t$, (43) and (44) converge to

$$\theta_{t'} + \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0\,, \tag{45}$$

and

$$x(s,t') = \int_0^s \cos\theta(s',t')ds' + x_0\,, \qquad v(s,t') = \int_0^s \sin\theta(s',t')ds' + v_0\,, \tag{46}$$

which give the (potential) WKI elastic beam equation (24).

3.2. A semi-discrete short pulse equation

We consider the semi-discrete sine-Gordon equation

$$\frac{d}{dt}(\theta_{n+1}-\theta_n) = 4\epsilon\sin\left(\frac{\theta_{n+1}+\theta_{n-1}}{2}\right). \tag{47}$$

Similar to the continuous case, the semi-discrete sine-Gordon equation (47) can be regarded as describing a certain motion of discrete plane curves. Therefore, we may expect that the application of the same transformation as the case of the semi-discrete WKI equation to the semi-discrete sine-Gordon equation (47) yields the semi-discrete analogue of the short pulse equation. By using the transformation

$$\gamma_n(t) = \begin{bmatrix} X_n(t) \\ v_n(t) \end{bmatrix} = \sum_{j=0}^{n-1} \begin{bmatrix} \epsilon \cos\left(\frac{\theta_{j+1}+\theta_j}{2}\right) \\ \epsilon \sin\left(\frac{\theta_{j+1}+\theta_j}{2}\right) \end{bmatrix} + \begin{bmatrix} X_0 \\ v_0 \end{bmatrix}, \quad (48)$$

we obtain the semi-discrete short pulse equation

$$\frac{d}{dt}(X_{n+1} - X_n) = -2(v_{n+1}^2 - v_n^2), \quad (49)$$

$$\frac{d}{dt}(v_{n+1} - v_n) = 2(X_{n+1} - X_n)(v_{n+1} + v_n). \quad (50)$$

We note that the following relation also holds from (48)

$$\left(\frac{v_{n+1} - v_n}{\epsilon}\right)^2 + \left(\frac{X_{n+1} - X_n}{\epsilon}\right)^2 = 1. \quad (51)$$

From (49) and (50), we obtain

$$\frac{d}{dt}\left(\frac{v_{n+1} - v_n}{X_{n+1} - X_n}\right) = 2(v_{n+1} + v_n) + 2\left(\frac{v_{n+1} - v_n}{X_{n+1} - X_n}\right)^2 (v_{n+1} + v_n). \quad (52)$$

In order to take the continuous limit, we assume the boundary condition $X_n = v_n = 0$ for $n < 0$, which is consistent with (48). Then the continuous limit $\epsilon \to 0$ (i.e., $X_{n+1} - X_n \to 0$) gives

$$\frac{v_{n+1} - v_n}{X_{n+1} - X_n} \to \frac{\partial v}{\partial x}, \quad \frac{v_{n+1} + v_n}{2} \to v,$$

$$\frac{\partial X_n}{\partial t} = \frac{\partial X_0}{\partial t} + \sum_{j=0}^{n-1} \frac{\partial (X_{j+1} - X_j)}{\partial t}$$

$$= \frac{\partial X_0}{\partial t} - 2\sum_{j=0}^{n-1}(v_{j+1}^2 - v_j^2) = -2v_n^2 \to \frac{\partial x}{\partial t} = -2v^2,$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} = \frac{\partial}{\partial t'} - 2v_n^2\frac{\partial}{\partial x} \to \frac{\partial}{\partial t'} - 2v^2\frac{\partial}{\partial x}.$$

Thus (52) converges to

$$(\partial_{t'} - 2v^2\partial_x)v_x = 4v + 4vv_x^2, \quad (53)$$

which is nothing but the short pulse equation (6). The same result was obtained by using the bilinear method [23].

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