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# Finite-time Lyapunov exponents in nonlinear dynamical systems with time-delayed feedback

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**Abstract–** Nonlinear dynamical systems with time-delayed feedback show very rich dynamics due to the high-dimensionality of the systems induced by time-delayed feedback. Dynamical systems generate chaotic and regular motions, which induce local instability. These local motions can be quantified with finite-time Lyapunov exponents (FTLEs). However, methods for the calculation of FTLEs in time-delayed dynamical systems have not been well established yet because of the difficulty of calculating Lyapunov exponents in time-delayed high-dimensional systems. We present a method for calculating FTLEs in time-delayed dynamical systems, and apply it to the Mackey-Glass model and the Lang-Kobayashi equations for a semiconductor laser with optical feedback. We investigate the distributions of FTLEs for different parameter values. It is found that both the variance of the distribution of FTLEs and the maximum Lyapunov exponent decrease with increase of the delay time in chaotic regions.

## 1. Introduction

Nonlinear dynamical systems with time-delayed feedback show bifurcation phenomena and chaotic behaviors which have high dimensionality due to the time-delayed feedback [1,2]. The time-delayed dynamical systems can be used for the applications of fast physical random number generators [3,4], reservoir computing [5,6], and optical secure communication.

The dimensionality of dynamical systems can be quantified from Lyapunov exponents. Lyapunov exponents characterize an average growth of small perturbations to an orbit of an attractor in the phase space and a positive maximum Lyapunov exponent denotes that the system is governed by deterministic chaos. Although the Lyapunov exponents are asymptotic quantities of an attractor, it may be necessary to investigate a growth rate of perturbations over finite time. The characteristics of finite-time behaviors are important for prediction of chaotic time series [7] and for unpredictability of chaos-based random number generators [8]. The finite-time behaviors are also useful to distinguish intermittent dynamical states from continuous chaotic motions [9].

To quantify a growth rate for finite-time behaviors, finite-time Lyapunov exponents (FTLEs) [9] and local Lyapunov exponents (LLEs) [10,11] have been proposed. The distributions of FTLEs in fully developed chaos and intermittency have been classified [9]. Local stability of an attractor in a system driven by an external signal has been investigated from conditional LLEs [11]. The FTLEs and LLEs have been used for the investigation of local stability in an attractor. However, methods for calculating FTLEs in time-delayed dynamical systems have not been well established yet. It is important to develop a method for calculating FTLEs in time-delayed dynamical systems.

In this study, we propose a method for calculating FTLEs in time-delayed dynamical systems, and apply the method to the Mackey-Glass model and the Lang-Kobayashi equations for a semiconductor laser with optical feedback. We investigate the dependence of the distributions of the FTLEs on system parameter values.

## 2. Numerical models

We used the Mackey-Glass equation in our numerical simulations. The Mackey-Glass equation is a model with time-delayed feedback and it is represented by the following equation,

$$\frac{dx(t)}{dt} = \frac{ax(t-\tau)}{1+x^b(t-\tau)} - cx(t) \qquad (1)$$

where $x(t-\tau)$ denotes the time-delayed feedback and $\tau$ is the delay time. The delay time is set to $\tau = 5$. The parameters $a$, $b$, and $c$ are $a = 2$, $b = 10$, and $c = 1$, respectively.

We also used the Lang-Kobayashi equations in our simulations. The dynamics of a single mode semiconductor laser subject to coherent optical feedback with time delay can be represented by the Lang-Kobayashi equations [12]. The Lang-Kobayashi equations consist of differential equations for slowly varying amplitude of the electric field $E$ (a complex variable) and the carrier density $N$ (a real variable):

$$\frac{dE(t)}{dt} = \frac{1+i\alpha}{2}\left[\frac{G_N\left(N(t)-N_0\right)}{1+\varepsilon|E(t)|^2}\right]E(t)$$

$$+\kappa E(t-\tau)\exp(-i\omega t) \qquad (2)$$

$$\frac{dN(t)}{dt} = J - \frac{N(t)}{\tau_s} - \frac{G_N\left(N(t)-N_0\right)}{1+\varepsilon|E(t)|^2}|E(t)|^2 \qquad (3)$$

where $E(t - \tau)$ denotes the time-delayed optical feedback and $\tau$ is the delay time, which is defined as $\tau = 2L / c = 4.0$ ns by the external cavity length $L$ and the speed of light $c$. $\kappa = (1 - r_2^2)\, r_3 / (r_2 \tau_{in}) = 6.2$ ns$^{-1}$ is the feedback strength, where $r_2$ and $r_3$ are the intensity reflectivities of the laser facet and the external mirror, and $\tau_{in}$ is the round-trip time in the internal laser cavity. $\alpha$ is the linewidth enhancement factor, $G_N$ is the gain coefficient, $N_0$ is the carrier density at the transparency, $\tau_p$ is the photon decay rate, $\omega$ is the optical angular frequency, $\varepsilon$ is the gain saturation coefficient, $J = 1.36 J_{th}$ is the injection current, $J_{th}$ is the injection current at lasing threshold, and $\tau_s$ is the carrier decay rate. We observe the intensity of the semiconductor laser output $I(t) = |E(t)|^2$.

In our numerical simulations, we integrated the model equations by using the forth-order Runge-Kutta method. The time-delayed feedback is discretized with an integration time step $h$. The time-delayed systems are treated as finite dimensional systems by this discretization.

## 3. Finite-time Lyapunov exponents in time-delayed dynamical systems

Dynamical systems with time-delayed feedback have high dimensionality and the construction of the phase space trajectory differs from the case for ordinary systems without time delay. We explain a method for calculating FTLEs in a dynamical system with time-delayed feedback.

Lyapunov exponents denote a growth rate of small perturbations. We consider a small deviation $\delta\boldsymbol{x}$ from an original chaotic trajectory $\boldsymbol{x}$. We can obtain linearized equations for $\delta\boldsymbol{x}$ by linearizing the original equations.

$$\frac{d\delta\boldsymbol{x}(t)}{dt} = \mathrm{J}[\boldsymbol{x}(t)]\delta\boldsymbol{x}(t) + \mathrm{J}_\tau[\boldsymbol{x}(t-\tau)]\delta\boldsymbol{x}(t-\tau) \qquad (4)$$

where J and $\mathrm{J}_\tau$ are the Jacobean with respect to $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t - \tau)$, $\tau$ represents delay time. It is necessary to calculate a norm $d(t)$ in state space to obtain Lyapunov exponents. All of the variables within delay time have been considered as components of a state vector in time-delayed dynamical systems. In numerical simulations, the variables within the delay time are discretized with an integration time step $h$, and the number of variables within the delay time $\tau$ is $M = \tau / h$. The norm can be calculated by using $M$ variables as following,

$$d(t) = \sqrt{\sum_{i=0}^{M-1} |\boldsymbol{x}(t-ih)|^2} \qquad (5)$$

The maximum Lyapunov exponent (MLE) of the dynamical system can be obtained by averaging the logarithm of fraction of the norm.

$$\lambda_{\max} = \frac{1}{\tau N}\sum_{i=0}^{N-1} \ln\frac{d(t+i\tau)}{d(t-(i+1)\tau)} \qquad (6)$$

where $N$ is the number of calculation step for normalization of the norm vector. $\lambda_{max}$ is an asymptotic value over an attractor.

We define a finite-time Lyapunov exponent (FTLE) using the definition of MLE in Eq. (6). The FTLE for the delay time $\tau$ is represented as follows:

$$\lambda_{\max} = \frac{1}{\tau}\ln\frac{d(t)}{d(t-\tau)} \qquad (7)$$

This equation can be transformed for the calculation of FTLE for the integration step $h$,

$$\begin{aligned}\lambda_\tau &= \frac{1}{\tau}\ln\left(\frac{d(t)}{d(t-h)}\times\frac{d(t-h)}{d(t-2h)}\times\cdots \times\frac{d(t-\tau+2h)}{d(t-\tau+h)}\times\frac{d(t-\tau+h)}{d(t-\tau)}\right)\\ &= \frac{1}{\tau}\left(\ln\frac{d(t)}{d(t-h)}+\ln\frac{d(t-h)}{d(t-2h)}+\cdots +\ln\frac{d(t-\tau+2h)}{d(t-\tau+h)}+\ln\frac{d(t-\tau+h)}{d(t-\tau)}\right)\\ &= \frac{1}{hM}\sum_{i=0}^{M-1}\ln\frac{d(t-ih)}{d(t-(i+1)h)} \qquad (8)\end{aligned}$$

where we use $\tau = hM$ and $M$ is an positive integer. Therefore, the FTLE for the integration time step $h$ (i.e., also known as the local Lyapunov exponent) can be defined as,

$$\lambda_{local} = \frac{1}{h}\ln\frac{d(t)}{d(t-h)} \qquad (9)$$

From these results, the FTLE for time duration of $T = hK$ ($K$ is an positive integer) can be defined as,

$$\begin{aligned}\lambda_{fin} &= \frac{1}{hK}\sum_{i=0}^{K-1}\ln\frac{d(t-ih)}{d(t-(i+1)h)}\\ &= \frac{1}{T}\ln\frac{d(t)}{d(t-T)} \qquad (10)\end{aligned}$$

We investigate probability distribution and its variance of $\lambda_{fin}(T)$ in the Mackey-Glass and Lang-Kobayashi models for various $T$. $T$ is the finite time for which the growth rate of perturbations is measured. Note that $\lambda_{fin}(T)$ approaches MLE for large $T$.

## 4. Numerical results of Mackey-Glass model

We used the integration time step $h = 0.05$ for the numerical simulations of the Mackey-Glass model. A chaotic temporal waveform of the Mackey-Glass model is shown in Fig. 1(a). The maximum Lyapunov exponent $\lambda_{max}$ of this waveform is 0.058, indicating that the dynamics is governed by deterministic chaos. We calculated the probability distributions of FTLEs for the temporal waveform of Fig. 1(a), as shown in Fig. 1(b). We used three values of the finite time $T$ (= 1, 5, and 50) to calculate FTLEs. For $T = 1$ (the black curve), a peak of the probability distribution is located at $\lambda_{fin}(T) = 0$. The distribution of positive values of $\lambda_{fin}(T)$ is larger than that of negative values of $\lambda_{fin}(T)$, which results in the positive MLE of $\lambda_{max} = 0.058$. For $T = 5$, the peak value of the probability distribution of $\lambda_{fin}(T)$ approaches $\lambda_{max}$ and the peak height becomes lower than that for $T = 1$. For larger $T = 50$, the peak height becomes higher and the distribution is narrower than that for $T = 5$.

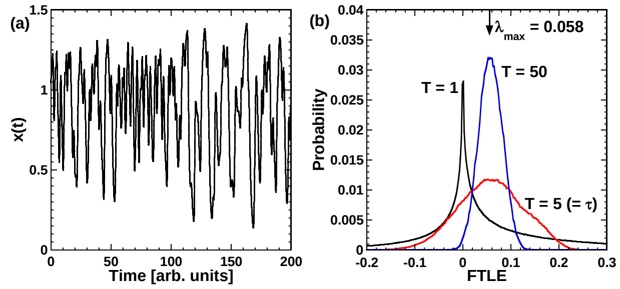


Fig. 1 (a) Temporal waveform of the Macke-Glass model and (b) probability distributions of the FTLEs for different finite time $T$ ($T$ = 1, 5, and 50). The parameter values of $a$ = 2, $b$ = 10, $c$ = 1, and $\tau$ = 5 are used.

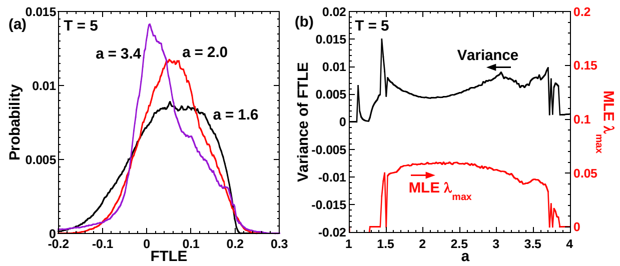


Fig. 2 (a) Probability distributions of the FTLEs for different parameter values of $a$ ($a$ = 1.6, 2.0, and 3.4). (b) Variance of the distributions of the FTLEs and the MLE as a function of $a$ in the Mackey-Glass model. The other parameter values are the same as in Fig. 1. $T = 5$ is used to calculate FTLEs.

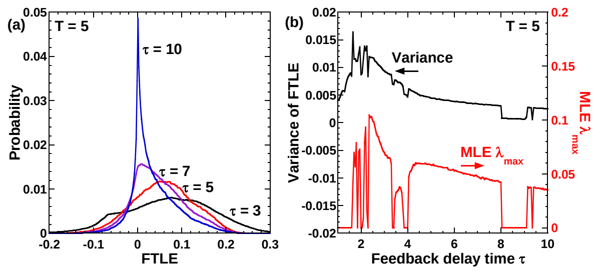


Fig. 3 (a) Probability distributions of the FTLEs for different parameter values of the delay time $\tau$ ($\tau$ = 3, 5, 7, and 10). (b) Variance of the distributions of the FTLEs and the MLE as a function of $\tau$ in the Mackey-Glass model. The other parameter values are the same as in Fig. 1. $T = 5$ is used to calculate FTLEs.

We investigate variances of the probability distributions of FTLEs when the parameter $a$ is varied in the Mackey-Glass model. Figure 2(a) shows the distributions of FTLEs for different $a$ ($a$ = 1.6, 2.0, and 3.4). We used $T = 5$, which equals to the delay time $\tau$, for the calculation of FTLEs. The MLEs are $\lambda_{max}$ = 0.050, 0.058, and 0.044 for the parameter values of $a$ = 1.6, 2.0, and 3.4, respectively. The peak value of the distribution becomes higher and the distribution becomes narrower with increase of $a$. We investigate the variances of FTLEs and the MLE as a function of $a$ as shown in Fig. 2(b). Chaotic dynamics are obtained in $1.44 < a < 3.80$ where positive $\lambda_{max}$ are obtained (the dashed red curve in Fig. 2(b)). The MLE has the maximum value at a~2.0, where the variance has a local minimum value. It is worth noting that the variance is increased as $\lambda_{max}$ is decreased for different values of $a$.

Next, we changed the delay time $\tau$ in the Mackey-Grass model. Figure 3(a) shows the probability distributions of FTLEs for different delay times $\tau$ = 3, 5, 7, and 10. We fixed the finite time $T = 5$ for the calculation of FTLEs. The distribution becomes sharper as the delay time is increased as shown in Fig. 3(a). We also investigated the variances and the MLE as the delay time is changed. Figure 3(b) shows the variance of FTLEs (the black solid curve) and the MLEs (the dashed red curve). Both the variance and MLE decrease with increase of the delay time in the chaotic regimes. Therefore, the change in the distribution of FTLEs is dependent of system parameters.

## 5. Numerical results of Lang-Kobayashi equations

Next we investigate FTLEs in the Lang-Kobayashi equations for a semiconductor laser with optical feedback. We used the integration time step $h$ = 0.005 ns in our numerical simulations. Figure 4(a) shows a temporal waveform of a semiconductor laser subject to time-delayed optical feedback with the feedback strength $\kappa$ = 6.21 $ns^{-1}$. The MLE for this temporal waveform is $\lambda_{max}$ = 0.82 $ns^{-1}$. The probability distributions of FTLEs for different $T$ are shown in Fig. 4(b). For small $T = 0.5$ ns (the black curve in Fig. 4(b)), the peak of the distribution is located at $\lambda_{fin} = 0$ $ns^{-1}$. The peak of the distribution is shifted from 0 to $\lambda_{max}$ and the variance of the distribution becomes smaller as $T$ is increased ($T$ = 4, 8, 64 ns in Fig. 4(b)). This behavior is similar to the case of the Mackey-Glass model shown in Fig. 1(b).

We investigate variances of probability distributions of FTLEs when the coupling strength $\kappa$ is varied. Figure 5(a) shows the probability distributions for different coupling strengths $\kappa$ = 4.66, 6.21, and 7.77 $ns^{-1}$. For $\kappa$ = 4.66 $ns^{-1}$, the distribution has a large peak. The peak of the distribution is shifted to the positive direction and becomes lower value with increase of $\kappa$. The variance of the distribution becomes larger and the probability of positive $\lambda_{fin}(T)$ increases with increase of $\kappa$. Figure 5(b) shows the variance (the black curve) and $\lambda_{max}$ (the red curve) as a function of $\kappa$. Both the variance and $\lambda_{max}$ increase monotonically with increase of $\kappa$ in the chaotic region at $\kappa$ > 4.4 $ns^{-1}$.

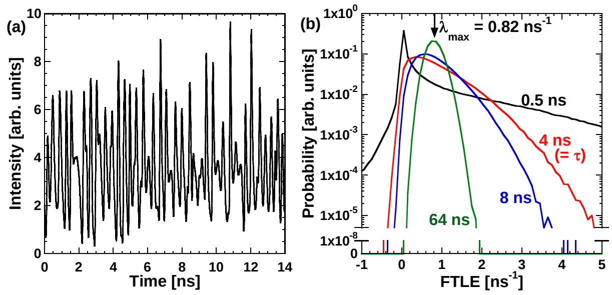


Fig. 4 (a) Temporal waveform of the Lang-Kobayashi equations. (b) Probability distributions of the FTLEs for different finite times $T$ ($T$ = 0.5, 4, 8, and 64 ns). The feedback strength $\kappa$ = 6.21 $ns^{-1}$ and the delay time $\tau$ = 4 ns are used. Note that the vertical axis in (b) is semi-logarithmic plot.

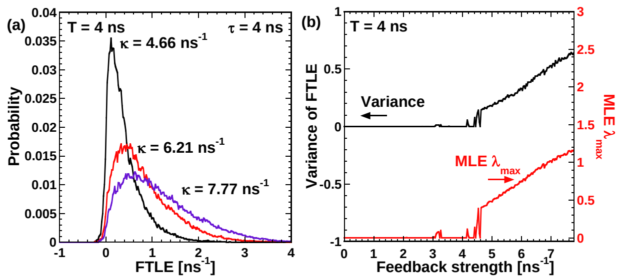


Fig. 5 (a) Probability distributions of the FTLEs for different parameter values of the feedback strength $\kappa$ ($\kappa$ = 4.66, 6.21, and 7.77 $ns^{-1}$). (b) Variance of the FTLEs and the MLE as a function of $\kappa$ for the fixed delay time ($\tau$ = 4 ns) in the Lang-Kobayashi model. $T$ = 4 ns is used to calculate FTLEs.

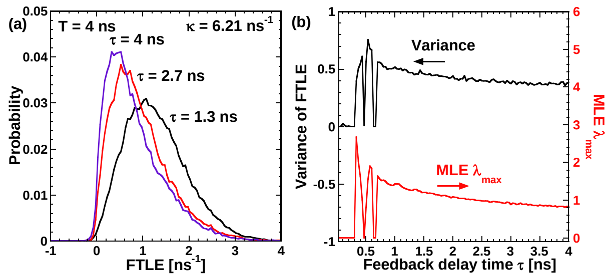


Fig. 6 (a) Probability distributions of the FTLEs for different parameter values of the delay time $\tau$ ($\tau$ = 1.3, 2.7, and 4.0 ns) (b) Variance of the FTLEs and the MLE as a function of $\tau$ for the fixed feedback strength ($\kappa$ = 6.21 $ns^{-1}$) in the Lang-Kobayashi model. $T$ = 4 ns is used to calculate FTLEs.

Figure 6(a) shows the probability distributions of FTLEs for different delay time $\tau$ ($\tau$ = 1.3, 2.7, and 4.0 ns) at the fixed feedback strength ($\kappa$ = 6.21 $ns^{-1}$). The peak of the distributions is shifted to the negative direction and the distribution becomes narrower with increase of $\tau$. This result is similar to that of the Mackey-Glass model shown in Fig. 3(a). Figure 6(b) shows the variance of FTLEs and the MLEs as a function of $\tau$. Both the variance (the black curve) and the MLEs (the red curve) decrease monotonically with increase of $\tau$. This result is similar to the case for the Mackey-Glass model as shown in Fig. 3(b).

## 6. Conclusion

We have proposed the method for calculating the FTLEs in time-delayed dynamical systems. Since time-delayed dynamical systems have high dimensionality due to time-delayed feedback, all the variables within the delay time need to be considered as independent variables that are used for the calculation of the norm vector for the linearized equations. We have applied this method to the Mackey-Glass model and the Lang-Kobayashi equations and investigated the dependence of the probability distributions of the FTLEs on some parameter changes. For both of the time-delayed systems the variance of the distribution of FTLEs and MLEs decrease with increase of the delay time in the chaotic regime.

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