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Existence and stability of discrete breathers in Fermi-Pasta-Ulam lattices

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Abstract—Discrete breathers are spatially localized periodic solutions in nonlinear lattices. We have proved the existence of two types of discrete breathers, i.e., the Sievers-Takeno and Page modes, in one-dimensional Fermi-Pasta-Ulam lattices, based on an approach using a fixed point theorem to the associated homogeneous potential lattice. Moreover, we have proved that the Sievers-Takeno mode is spectrally unstable while the Page mode is spectrally stable.

1. Introduction

Spatially localized excitations in nonlinear space-discrete dynamical systems have attracted great interest since the ground-breaking work by Takeno *et al.* [1, 2]. The localized modes are called *discrete breathers* (DBs) or *intrinsic localized modes*. The DB is a time-periodic and spatially localized solution of the equations of motion. It is expected that the DB is a quite general form of localized excitation emerging in a variety of nonlinear space-discrete systems in nature. Indeed, experimental evidence for the existence of DB has been reported in various systems [3, 4, 5, 6, 7]. Considerable progress has been achieved in understanding the nature of DB so far (e.g., [8, 9, 10, 11] and references therein).

From the theoretical point of view, fundamental issues are existence and stability of the DB solutions. Their existence has been proved in several nonlinear lattice models [12, 13, 14, 15, 16]. The anti-continuous limit, which was proposed by MacKay and Aubry [12], is the most useful concept for proving the existence of DBs. An approach based on this concept has been applied to various lattice models such as the nonlinear Klein-Gordon lattice [12], the nonlinear Schrödinger lattice [12], and the diatomic Fermi-Pasta-Ulam (FPU) lattice [13]. The stability of DBs has been mainly studied near the anti-continuous limit in several lattice models [17, 18, 19, 20, 21, 22, 23].

There are a number of lattice models, to which the anti-continuous limit approach is not applicable. Among them, a typical and important model is the monoatomic FPU lattice. It is known that there are two types of DBs having different spatial symmetries for this model, which are called the Sievers-Takeno (ST) mode [1, 2] and Page (P) mode [24]. These two modes were found by approximate analytical calculations.

For the FPU model, the existence proofs have been given only in the case of infinite size lattice by a homoclinic orbit method [14], a variational method [15], and a center manifold reduction method [16]. An existence proof for finite FPU lattices is still lacking, although the DBs are numerically observed in the finite lattices and this fact indicates that the infinite size is not essential for the existence of DB. In this paper, we present existence theorems of the ST and P modes for the one-dimensional FPU lattice with periodic boundary conditions. Our approach is different from those in Refs. [14, 15, 16]. A discrete breather solution is constructed in the associated homogeneous potential lattice by using a fixed point theorem and then it is continued to the nonhomogeneous potential one. An advantage of our approach is that it can provide a detail information on the DB profile, which cannot be obtained by the previous approaches.

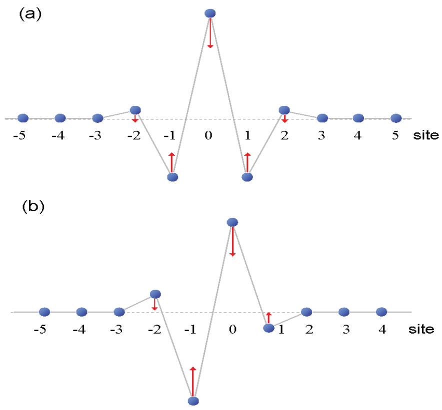

Figure 1: Profile of DB: (a) ST mode, and (b) P mode. Each particle oscillates out of phase with its nearest neighbors.

As for the stability of DB in the FPU lattice, it is only numerically shown that the ST and P modes are spectrally unstable and stable, respectively [25]. There has been no rigorous results on the spectral stability of DBs in the FPU lattice. Our approach is also advantageous from the point of view of stability analysis. The present theorems show that the ST mode is spectrally unstable while the P mode is spectrally stable.

2. Model

We consider the one-dimensional FPU lattice, which is described by the Hamiltonian

$$H = \sum_{i=-N'}^{N} \frac{1}{2} p_i^2 + \sum_{i=-N'}^{N} V(q_{i+1} - q_i), \tag{1}$$

where $q_i \in \mathbb{R}$ and $p_i \in \mathbb{R}$ represent the position and momentum of the ith particle, respectively. We employ the periodic boundary conditions, $q_{-(N'+1)} = q_N$ and $q_{N+1} = q_{-N'}$. We can assume $N' = N$ or $N - 1$ without loss of generality. The number N_0 of degrees of freedom of Hamiltonian (1) is given by $N_0 = N + N' + 1$. Hamiltonian (1) describes a one-dimensional chain of particles with nearest neighbour interactions. We suppose the interaction potential V of the form

$$V(X) = W(X, \mu) + \frac{1}{k} X^k, \tag{2}$$

where $k \geq 4$ is an even integer and $W(X, \mu) : \mathbb{R} \times O \to \mathbb{R}$ is a C^2 function of X and μ such that $W(X, 0) = 0$, where $\mu \in \mathbb{R}^l$ is a set of parameters and $O \subseteq \mathbb{R}^l$ is a neighbourhood of $\mu = 0$.

The equations of motion derived from Hamiltonian (1) are given by

$$\ddot{q}_i = V'(q_{i+1} - q_i) - V'(q_i - q_{i-1}), \tag{3}$$

where $i = -N', \ldots, N$. Let $\hat{\boldsymbol{q}}(t) = (\hat{q}_{-N'}(t), \ldots, \hat{q}_N(t))$ be a T-periodic solution of Eq. (3). The spectral stability of $\hat{\boldsymbol{q}}(t)$ is determined by the variational equations. Let ξ_i be the variation in q_i, and we denote as $\boldsymbol{\xi} = (\xi_{-N'}, \ldots, \xi_N)$. Linearizing Eq. (3) along $\hat{\boldsymbol{q}}(t)$, we obtain the variational equations in the vector form

$$\ddot{\boldsymbol{\xi}} + A(t)\boldsymbol{\xi} = 0, \tag{4}$$

where $A(t)$ is the Hessian matrix of the potential function evaluated on $\hat{\boldsymbol{q}}(t)$, i.e., its components are given by $A_{ij}(t) = \partial^2 \mathcal{V}(\hat{\boldsymbol{q}}(t))/\partial q_i \partial q_j$, where $\mathcal{V} = \sum_{i=-N'}^{N} V(q_{i+1} - q_i)$.

Let $\{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{2N_0}\}$ be a system of fundamental solutions of Eq. (4). According to the Floquet theory, the fundamental solutions of Eq. (4) at t and $t + T$ are related via a $2N_0 \times 2N_0$ monodromy matrix $\mathcal{M}$ as

$$\left(\boldsymbol{\xi}_1(t+T), \ldots, \boldsymbol{\xi}_{2N_0}(t+T)\right) = \left(\boldsymbol{\xi}_1(t), \ldots, \boldsymbol{\xi}_{2N_0}(t)\right) \cdot \mathcal{M}. \tag{5}$$

Eigenvalues of $\mathcal{M}$ are called the characteristic multipliers and they are independent of the choice of fundamental solutions. Let ρ_i, $i = 1, \ldots, 2N_0$ be the characteristic multipliers of $\hat{\boldsymbol{q}}(t)$. The spectral stability of $\hat{\boldsymbol{q}}(t)$ is defined as follows.

Definition 1. *Periodic solution $\hat{\boldsymbol{q}}(t)$ is said to be spectrally unstable if there exists ρ_i such that $|\rho_i| > 1$, otherwise it is said to be spectrally stable.*

3. Notations

In order to state our theorems in Sec. 4, we introduce some notations. Consider the differential equation

$$\ddot{\phi} + \phi^{k-1} = 0. \tag{6}$$

Let $\phi(t)$ be a solution of Eq. (6) with the initial conditions $\phi(0) = a > 0$ and $\dot{\phi}(0) = 0$. Equation (6) has the energy integral $\dot{\phi}^2/2 + \phi^k/k = h$, where $h > 0$ is an integration constant, and it is regarded as a Hamiltonian system with the potential ϕ^k/k. Since this potential is convex, it is clear that $\phi(t)$ is a non-constant periodic solution for any a. The period T of $\phi(t)$ depends on h $(= a^k/k)$, and it is obtained from the energy integral as follows:

$$T = 2\sqrt{2}\, h^{-(1/2-1/k)} \int_0^{k^{1/k}} \frac{1}{\sqrt{1 - x^k/k}}\, dx. \tag{7}$$

This indicates that T continuously varies from $T = +\infty$ to 0 as h varies from $h = 0$ to $+\infty$ since the integral in Eq. (7) is independent of h. This implies that for any given $T > 0$, there exists a non-constant periodic solution $\phi(t)$ with the period T. We denote this T-periodic solution of Eq. (6) with $\phi(t; T)$.

Let $\boldsymbol{x}$ be $\boldsymbol{x} = (x_{-N'}, \ldots, x_N) \in \mathbb{R}^{N_0}$, and S_{ST} and S_{P} be the subsets of $\mathbb{R}^{N_0}$ defined by

$$S_{\mathrm{ST}} = \{\boldsymbol{x};\ x_i = x_{-i},\ i = 1, 2, \ldots, N\}, \tag{8}$$

$$S_{\mathrm{P}} = \{\boldsymbol{x};\ x_i = -x_{-(i+1)},\ i = 0, 1, \ldots, N-1, \text{ and } x_N = 0 \text{ if } N' = N\}, \tag{9}$$

where $x_{-(N'+1)} = x_N$ in the case of $N' = N - 1$. These subsets S_{ST} and S_{P} are the subspaces of $\mathbb{R}^{N_0}$ which satisfy the spatial symmetries of ST and P modes, respectively (cf. Fig. 1).

We define a closed subset $B^m_{c,r} \subset \mathbb{R}^n$ as follows:

$$B^m_{c,r} = \Big\{\boldsymbol{x};\ |x_i| \leq c \text{ for } 0 \leq i \leq m,\ |x_i| \leq c r^{(k-1)^{i-m}} \text{ for } i > m\Big\}, \tag{10}$$

where $m \in \mathbb{N}$, $c > 0$, and $0 < r < 1$. This subset $B^m_{c,r}$ is specified by the three parameters (m, c, r). Equation (10) shows that the interval of x_i rapidly decreases with increasing i in $B^m_{c,r}$.

Consider the phase space $\mathbb{R}^{2N_0}$ of Hamiltonian system (1). We use the notations $\boldsymbol{q} = (q_{-N'}, \ldots, q_N)$ and $\boldsymbol{p} = (p_{-N'}, \ldots, p_N)$. Let Π_0 be the subspace defined by

$$\Pi_0 = \Big\{(\boldsymbol{q}, \boldsymbol{p});\ \textstyle\sum_{i=-N'}^{N} q_i = 0,\ \sum_{i=-N'}^{N} p_i = 0\Big\}. \tag{11}$$

This is the subspace in which both the mass center and the total momentum are zero.

4. Main results

Our main theorems for the existence and spectral stability of the DB solutions are stated as follows. Theorems 1 and 2 are for the ST and P modes, respectively.

Theorem 1. *Suppose that* $k = 4$*,* $N \geq 4$*, and* $T > 0$*. Let* $\boldsymbol{a} = (a_{-N'}, \ldots, a_N) \in \mathbb{R}^{N_0}$ *be a constant vector such that* $a_0 = 0.3762$*,* $a_{\pm 1} = -0.1968$*,* $a_{\pm 2} = 8.67 \times 10^{-3}$*, and* $a_i = 0$ *(otherwise). Then, there exists a unique* $\boldsymbol{x} \in B^m_{c,r} \cap S_{\mathrm{ST}}$ *with* $m = 3$*,* $c = 10^{-4}$*, and* $r = 3 \times 10^{-3}$ *such that* $\Gamma^0_{\mathrm{ST}}(t; T) : \boldsymbol{q} = \boldsymbol{u}\phi(t; T)$*,* $\boldsymbol{p} = \boldsymbol{u}\dot{\phi}(t; T)$ *is a* T*-periodic solution of FPU lattice (1) with* $\mu = 0$*, where* $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{x}$*. Moreover, there exist a neighbourhood* $U \subseteq \mathbb{R}^l$ *of* $\mu = 0$ *and a unique family* $\Gamma_{\mathrm{ST}}(t; T, \mu)$ *of* T*-periodic solutions of system (1) for* $\mu \in U$ *such that* $\Gamma_{\mathrm{ST}}(t; T, \mu)$ *is* C^1 *with respect to* t *and* μ*,* $\Gamma_{\mathrm{ST}}(t; T, \mu) \in \Pi_0$*, and* $\Gamma_{\mathrm{ST}}(t; T, 0) = \Gamma^0_{\mathrm{ST}}(t; T)$*. The periodic solution* $\Gamma_{\mathrm{ST}}(t; T, \mu)$ *is spectrally unstable with one unstable characteristic multiplier.*

Theorem 2. *Suppose that* $k = 4$*,* $N \geq 4$*, and* $T > 0$*. Let* $\boldsymbol{a} = (a_{-N'}, \ldots, a_N) \in \mathbb{R}^{N_0}$ *be a constant vector such that* $a_0 = a_{-1} = 0.323$*,* $a_1 = a_{-2} = -5.35 \times 10^{-2}$*, and* $a_i = 0$ *(otherwise). Then, there exists a unique* $\boldsymbol{x} \in B^m_{c,r} \cap S_{\mathrm{P}}$ *with* $m = 2$*,* $c = 3 \times 10^{-4}$*, and* $r = 6 \times 10^{-3}$ *such that* $\Gamma^0_{\mathrm{P}}(t; T) : \boldsymbol{q} = \boldsymbol{u}\phi(t; T)$*,* $\boldsymbol{p} = \boldsymbol{u}\dot{\phi}(t; T)$ *is a* T*-periodic solution of FPU lattice (1) with* $\mu = 0$*, where* $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{x}$*. Moreover, there exist a neighbourhood* $U \subseteq \mathbb{R}^l$ *of* $\mu = 0$ *and a unique family* $\Gamma_{\mathrm{P}}(t; T, \mu)$ *of* T*-periodic solutions of system (1) for* $\mu \in U$ *such that* $\Gamma_{\mathrm{P}}(t; T, \mu)$ *is* C^1 *with respect to* t *and* μ*,* $\Gamma_{\mathrm{P}}(t; T, \mu) \in \Pi_0$*, and* $\Gamma_{\mathrm{P}}(t; T, 0) = \Gamma^0_{\mathrm{P}}(t; T)$*. The periodic solution* $\Gamma_{\mathrm{P}}(t; T, \mu)$ *is spectrally stable.*

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