MONTE CARLO MODELLING OF OPTICAL RADIATION PROPAGATION IN STOCHASTIC SCATTERING MEDIA

Boris A. Kargin
Institute of Computational Mathematics and Mathematical Geophysics SB RAS,
Pr. Acad. Lavrentieva, 6, Novosibirsk, 630090, Russia
Novosibirsk state university, Pirogova str., 2, Novosibirsk, 630090, Russia
bkargin@osmf.sscc.ru

1. Introduction

To stochastic problems of transfer theory we refer the problems, where spatial variations of optical parameters of the scattering media are of random nature. The transfer of optical radiation in a substance may be described by integral equation [1]

$$f(\vec{x}) = \int_X k(\vec{x}', \vec{x}) f(\vec{x}') d\vec{x}' + \psi(\vec{x}),$$
$$k(\vec{x}', \vec{x}) = \frac{\Sigma_s(\vec{r}') g(\mu, \vec{r}') \exp(-\tau(\vec{r}', \vec{r})) \Sigma(\vec{r})}{2\pi |\vec{r} - \vec{r}'|^2 \Sigma(\vec{r}')} \delta(\omega - \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}), \quad (1)$$

where $f(\vec{x})$ is the collision density, $\vec{x} = (\vec{r}, \vec{\omega})$ and $\vec{x}' = (\vec{r}', \vec{\omega}')$ are the points of the phase space $X = \{\vec{r} \in R \subset R^3, \ \vec{\omega} = (a,b,c) \in \Omega = (a^2 + b^2 + c^2 = 1)\}$, $\mu = \left((\vec{\omega}', \vec{r} - \vec{r}') / |\vec{r} - \vec{r}'|\right)$ is the cosine of the scattering angle; $g(\mu, \vec{r})$ is the scattering phase function (scattering indicatrix) such that $\int_{-1}^{1} g(\mu, \vec{r}) d\mu = 1$; $\tau(\vec{r}', \vec{r}) = \int_0^l \Sigma(\vec{r}(s)) ds$ is the optical length of the segment $[\vec{r}', \vec{r}]$, $\vec{r}(s) = \vec{r}' + s(\vec{r} - \vec{r}')/l$, $l = |\vec{r}' - \vec{r}|$; $\Sigma(\vec{r}) = \Sigma_a(\vec{r}) + \Sigma_s(\vec{r})$ is the extinction coefficient of the flux, $\Sigma_a(\vec{r})$ is the absorption coefficient, $\Sigma_s(\vec{r})$ is the scattering coefficient; $\psi(\vec{x})$ is the distribution density of the source, $\int_X \psi(\vec{x}) d\vec{x} = 1$. Equation (1) determines the corresponding random Markov chain of collisions with the initial states $\psi(\vec{x})$ and the transition density $k(\vec{x}', \vec{x})$. The Monte Carlo method is in the simulation of this chain of trajectories on a computer and of the computation of statistical estimates for the sought for functionals. We consider a problem of computing the linear functionals $J_\varphi = (f, \varphi)$ from the solution of Eq.(1); here $\varphi(\vec{x}) \geq 0$. In the stochastic case, the kernel $k(\vec{x}', \vec{x})$ and the density of collisions $f(\vec{x})$ are dependent on a random field $\sigma(\vec{r}) = (\sigma_1(\vec{r}), \ldots, \sigma_s(\vec{r}))$, denoting the set of s optical parameters of the medium. The problem is solved on the basis of computation of some random values $\xi(\omega, \sigma)$ given on the trajectories ω of the simulated random process such that $E_{\omega(\sigma)}[\xi(\omega, \sigma)/\sigma] = J_\varphi(\sigma)$. Here $E_{\omega(\sigma)}$ denotes mathematical expectation with respect to the distribution of random trajectories ω depending on σ. Sought for functional is determined by $I_\varphi = \langle J_\varphi(\sigma) \rangle$, where $\langle\rangle$ denotes mathematical expectation with respect to a random field σ. The solution of the formulated stochastic problem by the Monte Carlo method is based on the principle of "double randomization'' resulting from the relation:

$\left\langle J_\varphi(\sigma)\right\rangle = \left\langle E_{\omega(\sigma)}[\xi(\omega,\sigma)/\sigma\right\rangle = E_{(\omega,\sigma)}\xi(\omega,\sigma).$ This relation shows that for the estimate J_φ it is sufficient to construct one trajectory ω for any realization of the random field σ. Thus, the problem to compute the functionals I_φ includes:

1) the construction of realizations of the random field σ;
2) simulation for each realization of σ of m conditionally independent trajectories of the Markov chain $(m \geq 1)$;
3) computation of corresponding random values $\xi(\omega,\sigma)$.

It is well known that for the majority of real transfer problems to increase the efficiency of the Monte Carlo calculations usually the weight algorithms are used [2]. In order to construct weight algorithms we use the Markov chain with some initial density $r_0(\vec{x})$ and transition density $r(\vec{x}', \vec{x}|\sigma)$ which should contain the generalized multiplier $\delta(\vec{\omega} - \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|})$ from Eq. (1). In this case one should calculate the auxiliary weights

$$Q_0(\vec{x}_0) = \frac{\psi(\vec{x}_0)}{r_0(\vec{x}_0)}, \qquad Q_n(\sigma) = Q_{n-1}(\sigma)\frac{k(\vec{x}_{n-1}, \vec{x}_n|\sigma)}{r(\vec{x}_{n-1}, \vec{x}_n|\sigma)} \tag{2}$$

at every collision, n - number of the state of random Markov chain (collision number). For the functional $J_\varphi(\sigma)$ the random estimate (so called "collision" estimate) $\xi(\sigma) = \sum_{n=0}^{N} Q_n(\sigma)\varphi(\vec{x}_n)$ is calculated, like that $J_\varphi(\sigma) = E_{\omega(\sigma)}\xi(\sigma)$, N – random number of the last state of the random trajectory $\omega(\sigma)$ (last collision before escape from the scattering medium).

2. Optimization of the Monte Carlo algorithm

Let us consider a problem to compute the functional I_φ for optical radiation scatted by the random inhomogeneous layer, in general, with three-dimensional continuous stochasticity, i.e. when $\Sigma(\vec{r})$ is a random field $(\vec{r} \in R = (-\infty, +\infty)^2 \times [0, H])$. It means that $\sigma(\vec{r}) = \Sigma(\vec{r})$ and all the rest optical parameters are known and determined. In this case we have two main problems. The first one is to construct a most adequate mathematical model of a stochastic layer and its efficient numerical implementation. In very many cases the sample size of numerical realizations of random field $\Sigma(\vec{r})$ should be statistically representative and the number of realizations may be compared to the number of trajectories. The second difficulty arises in simulation of photon trajectories in a random inhomogeneous 3D – medium. These both procedures take much computer time. To avoid these complications we suggest the following algorithm based on the well-known "dependent trials" method [1]. The given method's main idea is that the estimates of the sought for functional I_φ for different values of $\Sigma(\vec{r})$ may be obtained from the same random photon trajectories, using weights (2) to remove the appearing bias. Specifically, the trajectories constructed for $\Sigma(\vec{r}) = \Sigma_0(\vec{r})$ may be used to estimate the sought for functional for other values of $\Sigma(\vec{r})$, if after every transition $\vec{x}' \to \vec{x}$ the auxiliary weight of the particle is multiplied by the value $k(\vec{x}', \vec{x}|\Sigma)/r(\vec{x}', \vec{x}|\Sigma_0)$. Let $\omega_n = \{(\vec{x}_0, \vec{x}_1, ..., \vec{x}_n); \quad \vec{x}_i = (\vec{r}_i, \vec{\omega}_i), \quad i = \overline{0, n}\}$ be an arbitrary n-link trajectory, constructed with the transition density $r(\vec{x}', \vec{x}|\sigma) = k(\vec{x}', \vec{x}|\Sigma_0)$. Then one can easily see, that the weight multiplier $Q_n(\Sigma)$ corresponding to the realization of $\Sigma(\vec{r})$ is calculated by the formula

$$Q_n(\Sigma)=\prod_{i=1}^{n-1}\left\{\frac{\Sigma_s(\vec{r}_{i-1})\Sigma_0(\vec{r}_{i-1})}{\Sigma(\vec{r}_{i-1})\Sigma_{s,0}(\vec{r}_{i-1})}\right\}\cdot\frac{\Sigma(\vec{r}_n)}{\Sigma_0(\vec{r}_n)}e^{-\sum_{i=1}^{n}[\tau(\vec{r}_{i-1},\vec{r}_i)-\tau_0(\vec{r}_{i-1},\vec{r}_i)]}, \quad (3)$$

where $\tau_0(\vec{r}_{i-1},\vec{r}_i)=\int_0^{|\vec{r}_{i-1}-\vec{r}_i|}\Sigma_0(\vec{r}_{i-1}+s\vec{\omega}_{i-1})ds$, $\tau(\vec{r}_{i-1},\vec{r}_i)=\int_0^{|\vec{r}_{i-1}-\vec{r}_i|}\Sigma(\vec{r}_{i-1}+s\vec{\omega}_{i-1})ds$. Let $\xi(\Sigma)$ be the random estimate of the functional $J_\varphi(\Sigma)$ for the given realization of the random field $\Sigma(\vec{r})$. Then from (3) we can obtain that, in order to calculate $\xi(\Sigma)$ from the trajectory constructed with the transition density $k(\vec{x}',\vec{x}|\Sigma_0)$, it isn't necessary to construct the random field $\Sigma(\vec{r})$ at every point of the considered space, but we have to know the values of the random field $\Sigma(\vec{r})$ only at the points $\vec{r}_1,...,\vec{r}_N$, i.e. $\Sigma(\vec{r}_1),...,\Sigma(\vec{r}_N)$, as well as the values $\tau(\vec{r}_{i-1},\vec{r}_i)$, along the directions $\vec{\omega}_i=(\vec{r}_i-\vec{r}_{i-1})/|\vec{r}_i-\vec{r}_{i-1}|$ on the segments $(\vec{r}_{i-1},\vec{r}_i)$, $i=\overline{1,N}$.. Thus, the problem of calculating the functional $\langle J_\varphi(\Sigma)\rangle$, instead of averaging a random functional $J_\varphi(\Sigma)$ over the realizations of the random field $\Sigma(\vec{r})$, is reduced to determining it by the realizations of random vectors $\{\Sigma(\vec{r}_i)\}_{i=\overline{1,N}}$ and $\{\tau(\vec{r}_{i-1},\vec{r}_i)\}_{i=\overline{1,N}}$. In many cases this allows to considerably decrease the computing time. The effectiveness of such an approach is dependent on the model of the random field $\Sigma(\vec{r})$. To illustrate this approach, we shall limit ourselves to considering the following problem, related to the problem of optical radiation propagation through the continuous stratus-type stochastic cloudiness.

3. Numerical experiment

Suppose the scattering substance occupies the space $R=(-\infty,+\infty)\times(-\infty,+\infty)\times[0,H]$ and all optical characteristics of the medium are not dependent on the horizontal coordinates, $\Sigma(\vec{r})=\Sigma(z)$ be a random process with correlation function $K_\Sigma(s)$ and marginal distribution $p(\Sigma)$ with mean value $\overline{\Sigma}(z)$ and variance σ_Σ^2. One of the most popular spectral model of the random process $\Sigma(z)$ is an approximate model (see, for example, [3])

$$\Sigma(z)\approx\Sigma^{(k)}(z)=\overline{\Sigma}(z)+\sigma_\Sigma\sum_{j=1}^{k}a_j\sqrt{-2\ln\alpha_j}\cos(\lambda_j z+2\pi\beta_j),$$

where α_j and β_j are independent random values uniformly distributed in [0,1], $a_j^2=\frac{1}{k}$, λ_j are distributed on $[0,\infty)$ with probability density $s(\lambda)=\frac{2}{\pi}\int_0^\infty\cos(\lambda z)K(z)dz$. In this case $\tau[\vec{r}_{i-1},\vec{r}_i]$, $i=1,\ldots,n$ in (3) are calculated as follows

$$\tau[\vec{r}_{i-1},\vec{r}_i]=\frac{1}{|(\vec{\omega}_{i-1},\vec{k})|}\{\overline{\Sigma}|z_{i-1}-z_i|+\sigma_\Sigma\sum_{j=1}^{k}\frac{a_j}{\lambda_j}\sqrt{-2\ln\alpha_j}\times$$
$$\times[\sin(\lambda_j z_{i-1}+2\pi\beta_j)-\sin(\lambda_j z_i+2\pi\beta_j)]\}.$$

Here $\vec{k}=(0,0,1)$. In the capacity of the correlation function was used the following function $K_\Sigma(s)=\exp(-\tau s/H)$, where $\tau=\int_0^H\overline{\Sigma}(z)dz$ denotes the optical thickness of the determinate layer. The marginal distribution $p(\Sigma)$ of the random field $\Sigma(z)$ in accordance with [4] was chosen as $p(\Sigma)=(H/\tau)\exp(-H\Sigma/\tau)$. To illustrate the influence of stochasticity we compare functionals

from the solution of the transfer equation in a determinate flat cloud layer $0 \le z \le H$ with average values of those functionals corresponding to stochastic cloud layer when $\Sigma(z)$ is mentioned above random process. In the Table 1 the calculation results of the probability of passage through the scattering layer of radiation incident perpendicularly to the layer's upper boundary $z = 0$ are given, obtained by the described algorithm. The phase function $g(\mu, \vec{r}) = (1-\mu_0)/2 + \mu_0 \delta(\mu - 1)$ with an average cosine of the angle of scattering $\mu_0 = 0.9$ was considered in the capacity of the scattering indicatrix. The probability of survival of a photon in a collision (albedo of single scattering) Σ_s / Σ was set to be 0.7.

Table 1: Probabilities of radiation passage through the determinate and stochastic scattering layers

Layers	P_S, P_D	$\varepsilon, \%$	Ratio P_S / P_D	τ
S D	0.082 0.039	1.2 1.8	2.1	10
S D	0.022 0.0075	2.2 3.1	2.9	15
S D	0.0059 0.0015	3.3 3.8	3.9	20

Note: S – stochastical layer, D – determinate layer, P_S and P_D – probabilities of passage respectively through stochastic and determinate layers, ε - relative statistical error of the estimate for passage.

4. Conclusion

Due to the space limitation of the current paper it is not possible to carry out a more detailed analysis of, for example, the influence of 3D- stochasticity of the scattering medium on the transfer of optical radiation. But even the abovementioned calculation results for a simple enough model show that taking stochasticity into account gives an increase of the probability of passage. The ratio of the probability of passage in a stochastical medium to the probability of passage in a determinate medium grows with the increase of optical thickness. The obtained numerical results demonstrate that, when interpreting physical changes of radiation fields in a cloudy medium in some cases the stochastical properties of the medium play a significant role. The weight algorithm considered in this paper allows avoiding the laborious procedure of modeling random photon trajectories in randomly-inhomogeneous scattering media and therefore reducing the calculation time. Realization of the algorithm is quite simple. The profit in terms of calculation time depends upon the choice of model for the random field.

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