Space-Time Discretization of Maxwell's Equations in the Setting of Geometric Algebra

Mariusz Klimek #1, Ulrich Römer *2, Sebastian Schöps *,#3 , Thomas Weiland *4,#

Graduate School of Computational Engineering, Technische Universitaet Darmstadt
Dolivostrasse 15, D-64293, Darmstadt, Germany
[1] klimek@gsc.tu-darmstadt.de
[3] schoeps@gsc.tu-darmstadt.de

* *Institut für Theorie Elektromagnetischer Felder, Technische Universitaet Darmstadt*
Schlossgartenstrasse 8, D-64289 Darmstadt, Germany
[2] roemer@temf.tu-darmstadt.de
[4] thomas.weiland@temf.tu-darmstadt.de

***Abstract*—The Geometric Algebra (GA) for Minkowski space-time and Maxwell's equations in the setting of GA are briefly outlined. The constitutive equations are discussed in more detail. A discrete version of GA for a Cartesian grid is investigated and is shown to be equivalent to Tonti's approach. Furthermore, under quite natural assumptions both schemes coincide with the Finite Integration Technique (in 3D space) and Leap-Frog time integration.**

I. Introduction

The concept of Geometric Algebra (GA) was proposed by D. Hestenes, see e.g. [1], [2], and since then it was successfully applied in various areas of physics, mathematics and computer science [3], [4], [5]. This paper presents a brief introduction to GA. We focus on a special case, namely, the GA of 4D space-time. The use of geometric calculus is suppressed in order to stick to traditional calculus as close as possible.

A. Geometric Algebra of Minkowski Space-Time

In GA the fundamental operation (besides addition) is the *geometric product*, which in general is (where a, b and c are vectors) [3]

$$ab \neq ba, \qquad \text{(anticommutative)}$$
$$(ab)c = a(bc), \qquad \text{(associative)}$$
$$a(b+c) = ab + ac, \qquad \text{(left-distributive)}$$
$$(a+b)c = ac + bc, \qquad \text{(right-distributive)}$$
$$a^{-1} = \frac{a}{a^2}. \qquad \text{(invertible)}$$

A norm is defined through the relation $|a|^2 := a^2$, where a^2 is a scalar. Furthermore, the geometric product can be decomposed into its symmetric and antisymmetric part

$$ab = \frac{1}{2}(ab+ba) + \frac{1}{2}(ab-ba) := a\cdot b + a\wedge b,$$

where $\cdot$ and $\wedge$ denote the *scalar* and *exterior* products respectively. It is assumed that all scalars are real. In our terminology they are called 0-*vectors*.

4D basis vectors are denoted by γ_i. The convention is that $\gamma_t^2 = +1$ and $\gamma_x^2 = \gamma_y^2 = \gamma_z^2 = -1$. One important result is that orthogonal vectors, i.e., vectors a and b for which $a\cdot b = 0$ holds, anticommute. Note that a vector a is usually interpreted as an "arrow", i.e., an oriented 1D object.

The multiplication of two vectors leads to objects like $\gamma_t\gamma_x$, which are neither scalars, nor vectors, but *bivectors*. We interpret them as oriented 2D objects ("surfaces"). *Trivectors* represent oriented 3D objects ("volumes") and an example is $\gamma_t\gamma_x\gamma_z$.

The *pseudoscalar* $I := \gamma_t\gamma_x\gamma_y\gamma_z$ gives a unique representation of any 4-*vector* A_4 through $A_4 = |A_4|I$. We interpret I as an oriented 4D object ("space-time volume"), with square $I^2 = -1$; similar to the well-known property of the imaginary unit.

In space-time there are only 4 basis vectors, hence it is not possible to construct any 5-*vector*. Therefore, our algebra is now complete.

3D vectors $\sigma_k = \gamma_k\gamma_t$, $k = x, y, z$, square to $+1$ and are mutually orthogonal. Therefore, they behave like basis vectors of 3D space, but still (implicitly) contain temporal information. Additionally, $I = \sigma_x\sigma_y\sigma_z$ holds. We denote 3D vectors by an arrow, e.g., $\vec{E} \equiv E^x\sigma_x + E^y\sigma_y + E^z\sigma_z$.

B. Maxwell's Equations

We combine the charge density ϱ and the components $j^k, k = x, y, z$, of the electric current density $\vec{J}$ into a current density vector $J \equiv \varrho\gamma_t + j^x\gamma_x + j^y\gamma_y + j^z\gamma_z$. Note that $\vec{J}$ does not equal J with dropped temporal component - they are related via $J\gamma_t = \varrho + \vec{J}$. The fields $\vec{E}$ and $\vec{B}$ are combined to the Faraday bivector, e.g., [3]

$$F := \vec{E} + I\vec{B},$$

and $\vec{D}$ and $\vec{H}$ to

$$G := \vec{D} + I\vec{H}.$$

While the differential form of Maxwell's equation remains similar in the GA of space-time, the integral form differs. We denote by Ω a 3D region in space-time, and its boundary by $\partial\Omega$. Let V, S be a volume and a surface in space respectively. Applying the Fundamental Theorem of Geometric Calculus, [3] we obtain

$$\int_{\partial\Omega} (d^2x)\cdot F = 0, \quad (1)$$

$$\int_{\partial\Omega} (d^2x) \wedge G = \int_{\Omega} (d^3x) \wedge J, \quad (2)$$

which differs from

$$\begin{aligned} \int_{\partial S} \vec{E} \cdot d\vec{l} &= -\int_{S} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}, \\ \int_{\partial V} \vec{B} \cdot d\vec{S} &= 0, \end{aligned} \quad (3)$$

and

$$\begin{aligned} \int_{\partial S} \vec{H} \cdot d\vec{l} &= -\int_{S} \left(\frac{\partial \vec{D}}{\partial t} + \vec{J}\right) \cdot d\vec{S}, \\ \int_{\partial V} \vec{D} \cdot d\vec{S} &= \int_{V} \varrho dV. \end{aligned} \quad (4)$$

in the sense that (1)-(2) are integral equations in space-time, whereas in (3)-(4) we have to integrate in space and to differentiate in time.

C. Constitutive Equations

In this paper we will assume that the conductivity σ, permittivity ε and the permeability μ are scalar functions of space. Additionally, for simplicity, polarization and magnetization of the medium are assumed to be zero. Therefore the simplified constitutive equations are given by

$$\begin{aligned} \vec{D} &= \varepsilon \vec{E}, \\ \vec{H} &= \mu^{-1} \vec{B}, \end{aligned} \quad (5)$$

that are supplemented by Ohm's Law in lossy media

$$\vec{J} = \sigma \vec{E}. \quad (6)$$

The translation of the simplified constitutive equations (5) into the space-time form of GA reads

$$G = \frac{1}{2}\left\{(\varepsilon + \mu^{-1})F - (\varepsilon - \mu^{-1})\gamma_0 F \gamma_0\right\}, \quad (7)$$

where γ_0 is the four-velocity of the electric medium [3].

II. The Cartesian Grid

For discretization of Maxwell's equation in the GA setting we propose a finite-difference-like approach on a Cartesian grid in space-time, i.e., a natural generalization of Yee's scheme [6]. We will show that this general approach is equivalent to Tonti's space-time approach proposed in [7] and equivalent, up to minor modifications, to Finite Integration Technique (FIT) with Leap-Frog time-integration [8], [9].

The nodes of the primal mesh are introduced as

$$r^p_{i,j,k,l} := (i\Delta t, j\Delta x, k\Delta y, l\Delta z), \quad (8)$$

where i, j, k, l are integers. We further introduce edges, e.g., in x-direction we obtain

$$l^p_{i,j+1/2,k,l} := i\Delta t \times [j\Delta x, (j+1)\Delta x] \times k\Delta y \times l\Delta z, \quad (9)$$

that are oriented by the vector γ_x in $x-$direction. For simplicity we omit the respective definitions for the other coordinate directions. Facets are given by

$$\begin{aligned} &a^p_{i,j,k+1/2,l+1/2} := \\ &i\Delta t \times j\Delta x \times [k\Delta y, (k+1)\Delta y] \times [l\Delta z, (l+1)\Delta z], \end{aligned} \quad (10)$$

with orientation defined by the bivector $\gamma_y\gamma_z$. Volumes in 3D and space-time volumes are defined in a similar way. The midpoint of any space-time volume, is identified with a point on the dual grid. We then proceed as explained above to obtain edges, facets, volumes and space-time volumes on the dual grid respectively.

The Cartesian grid obtained in this way is an example of a dual orthogonal grid pair. This is the main reason for the efficiency of the Finite Difference Time Domain (FDTD) method and FIT [6], [8]: the discretization of the material parameters yields diagonal matrices.

III. Discrete Quantities

Equations (1) and (2) are discretized on the primal and dual grid respectively. We discretize (1) by restricting Ω to the 3D elements of the primal grid. Therefore we naturally associate to F the degrees of freedom

$$f_{i+1/2,j+1/2,k,l} := \int_{a^p_{i+1/2,j+1/2,k,l}} (d^2x) \cdot F. \quad (11)$$

The discretization of (2) on the dual grid leads to the degrees of freedom

$$g_{i+1/2,j+1/2,k,l} := I^{-1} \int_{a^d_{i+1/2,j+1/2,k,l}} (d^2x) \wedge G. \quad (12)$$

A link between quantities on the primal and dual grid can be established through the material relation (7). Therefore, we approximate (11) and (12) with a midpoint quadrature rule to obtain

$$\begin{aligned} f_{\overline{i+1/2},\overline{j+1/2},k,l} &\approx \big((|a^p|\gamma_0\gamma_x) \cdot F\big)(M^p_a) =: A^p \tilde{F}, \\ g_{i+1/2,j+1/2,\bar{k},\bar{l}} &\approx \big((|a^d|\gamma_y\gamma_z) \wedge G\big)(M^d_a) I^{-1} =: A^d \tilde{G} I^{-1}, \end{aligned}$$

where M^p_a and M^d_a are the centers of the primal and dual facets, respectively. For simplicity we omitted the indices on the right-hand-side. The bar over an index denotes integration over one mesh step in the corresponding dimension. Note that the quantities on the right-hand-side must be evaluated at the midpoints of the respective surfaces even if not stated explicitly. Since the geometric product is invertible we obtain the relations

$$\tilde{F} = (A^p)^{-1} f_{\overline{i+1/2},\overline{j+1/2},k,l}, \tag{13}$$

$$\tilde{G} = (A^d)^{-1} g_{i+1/2,j+1/2,\overline{k},\overline{l}} I, \tag{14}$$

for the bivectors $\tilde{F}, \tilde{G}$. We finally can use (7) to obtain

$$g_{i+1/2,j+1/2,\overline{k},\overline{l}} I = A^d \xi \left((A^p)^{-1} \right) f_{\overline{i+1/2},\overline{j+1/2},k,l}, \tag{15}$$

where we introduced the abbreviation:

$$\xi\left(A\right) := \frac{1}{2} \left[(\bar{\varepsilon} + \bar{\mu}^{-1}) A - (\bar{\varepsilon} - \bar{\mu}^{-1}) \gamma_0 A \gamma_0 \right], \tag{16}$$

where $\bar{\varepsilon}$, $\bar{\mu}$ are facet-averaged material parameters. Furthermore we assume that $\gamma_0 = \gamma_t$, i.e., the medium is at rest in the chosen coordinate system.

Finally, we obtain by (15) a diagonal discrete material matrix due to the fact that the dual grid introduced above gives an orthogonal one-to-one correspondence between primal and dual facets.

IV. Discretization

We now state the discrete versions of (1) and (2) explicitly. The degrees of freedom f, g naturally give rise to the degrees of freedom e, d, b, h for the electric field strength/flux density and the magnetic field strength/flux density respectively. E.g., we obtain

$$e_{\overline{i+1/2},\overline{j+1/2},k,l} := \int_{i\Delta t}^{(i+1)\Delta t} \int_{j\Delta x}^{(j+1)\Delta x} E^x|_{y=k\Delta y, z=l\Delta z} dx\, dt,$$

$$d_{i+1/2,j+1/2,\overline{k},\overline{l}} := \int_{(k-1/2)\Delta y}^{(k+1/2)\Delta y} \int_{(l-1/2)\Delta z}^{(l+1/2)\Delta z} D^x|_{t=(i+1/2)\Delta t, x=(j+1/2)\Delta x} dz\, dy,$$

and so on. Furthermore, with a bar under an index we denote a backward difference, i.e.,

$$h_{\overline{i},j+1/2,\underline{k+1/2},\overline{l}} := h_{\overline{i},j+1/2,k+1/2,\overline{l}} - h_{\overline{i},j+1/2,k-1/2,\overline{l}}.$$

Now, we can write down a discrete version of Maxwell's equations, i.e., Maxwell's grid equations in 4D. Since a 4D-box has 4 different types of 3D-facets, we will obtain 4 types of equations.

For the integration over a $x, y, z-$cube in (2), we obtain

$$\varrho_{i+1/2,\overline{j},\overline{k},\overline{l}} = d_{i+1/2,\underline{j+1/2},\overline{k},\overline{l}} + d_{i+1/2,\overline{j},\underline{k+1/2},\overline{l}} + d_{i+1/2,\overline{j},\overline{k},\underline{l+1/2}},$$

which is the same result as for FIT and Tonti's approach. However, for a $t, y, z-$cube in (2) we recover Tonti's discrete $x-$component of Faraday's Law

$$j_{\overline{i},j+1/2,\overline{k},\overline{l}} = h_{\overline{i},j+1/2,\underline{k+1/2},\overline{l}} - h_{\overline{i},j+1/2,\overline{k},\underline{l+1/2}} - d_{\underline{i+1/2},j+1/2,\overline{k},\overline{l}}, \tag{17}$$

which is also equivalent to the corresponding FIT equation

$$\Delta t\, j_{i,j+1/2,\overline{k},\overline{l}} = \Delta t\, h_{i,j+1/2,\underline{k+1/2},\overline{l}} - \Delta t\, h_{i,j+1/2,\overline{k},\underline{l+1/2}} - d_{\underline{i+1/2},j+1/2,\overline{k},\overline{l}} \tag{18}$$

if integration with respect to time is replaced by the midpoint quadrature rule. This is also true for the remaining equations.

Since we are using orthogonal primary/dual grids we obtain from (5) the discrete material relations

$$\frac{d_{i+1/2,j+1/2,\overline{k},\overline{l}}}{\Delta y \Delta z} = \bar{\varepsilon}^x \frac{e_{\overline{i+1/2},\overline{j+1/2},k,l}}{\Delta t \Delta x}, \tag{19}$$

$$\frac{b_{i,j,\overline{k+1/2},\overline{l+1/2}}}{\Delta y \Delta z} = \bar{\mu}^x \frac{h_{\overline{i},\overline{j},k+1/2,l+1/2}}{\Delta t \Delta x}, \tag{20}$$

where the material parameters are averaged in space (without time dependency), e.g., $\bar{\varepsilon}^x = \varepsilon_{\cdot,j+1/2,\overline{k},\overline{l}}/(\Delta y \Delta z)$. This corresponds to the classical averaging technique as used in FIT [9]. The FIT material laws (with the same averaging) read

$$\frac{d_{i+1/2,j+1/2,\overline{k},\overline{l}}}{\Delta y \Delta z} = \bar{\varepsilon}^x \frac{e_{i+1/2,\overline{j+1/2},k,l}}{\Delta x},$$

$$\frac{b_{i,j,\overline{k+1/2},\overline{l+1/2}}}{\Delta y \Delta z} = \bar{\mu}^x \frac{h_{i,\overline{j},k+1/2,l+1/2}}{\Delta x}. \tag{21}$$

Using the simplified material relations, we can express equations on the dual grid in terms of e and b variables. For example, using (20) in (17) gives

$$j_{\overline{i},j+1/2,\overline{k},\overline{l}} = \frac{\Delta t \Delta z}{\Delta x \Delta y} \frac{1}{\bar{\mu}^z} b_{i,\overline{j+1/2},\underline{\overline{k+1/2}},l} - \frac{\Delta t \Delta y}{\Delta x \Delta z} \frac{1}{\bar{\mu}^y} b_{i,\overline{j+1/2},k,\underline{\overline{l+1/2}}} + \frac{\Delta y \Delta z}{\Delta x} \bar{\varepsilon}^x \frac{e_{\underline{\overline{i+1/2}},\overline{j+1/2},k,l}}{\Delta t},$$

whereas using (21) in (18) we obtain its FIT equivalent

$$\Delta t j_{i,j+1/2,\overline{k},\overline{l}} = \frac{\Delta t \Delta z}{\Delta x \Delta y} \frac{1}{\bar{\mu}^z} b_{i,\overline{j+1/2},\underline{\overline{k+1/2}},l} - \frac{\Delta t \Delta y}{\Delta x \Delta z} \frac{1}{\bar{\mu}^y} b_{i,\overline{j+1/2},k,\underline{\overline{l+1/2}}} + \frac{\Delta y \Delta z}{\Delta x} \bar{\varepsilon}^x e_{\underline{i+1/2},\overline{j+1/2},k,l}.$$

Note that we obtain $e_{\overline{i+1/2},\overline{j+1/2},k,l}/\Delta t$ and $j_{\overline{i},j+1/2,\overline{k},\overline{l}}$ instead of the FIT quantities $e_{i+1/2,\overline{j+1/2},k,l}$ and $\Delta t j_{i,j+1/2,\overline{k},\overline{l}}$ respectively.

One can verify that the analogous statement to the one presented above is true for all other (primal and dual) equations and Ohm's law.

Although a direct application of Ohm's law $\vec{J} = \sigma \vec{E}$ is impossible, because the degrees of freedom for the electric current and electric field are located at different times. Therefore, we have to interpolate one of them in time. We choose to follow Tonti's approach, [7], i.e., we interpolate e and obtain

$$j_{\overline{i},j+1/2,\overline{k},\overline{l}} = \frac{\bar{\sigma}^x}{2} \left(e_{\overline{i+1/2},\overline{j+1/2},k,l} + e_{\overline{i-1/2},\overline{j+1/2},k,l} \right),$$

from which we obtain by midpoint quadrature the classical FIT version

$$j_{i,j+1/2,\overline{k},\overline{l}} = \frac{\bar{\sigma}^x}{2} \left(e_{i+1/2,\overline{j+1/2},k,l} + e_{i-1/2,\overline{j+1/2},k,l} \right).$$

without the integration over time.

V. Relation to Leap-Frog Time Integration

We disregard losses and consider in the following only the external source current $\vec{J}_e$. Nonetheless the treatment of Ohm's Law is straightforward and it is only supressed for simplicity of notation.

Furthermore we adopt the same enumeration scheme for degrees of freedom as in FIT [9] and suppress the spatial part of the indices to keep formulae short. Therefore, e.g., $\mathbf{b}^i$ is a vector of degrees of freedom at time $t = i\Delta t$ such as $b_{i,\overline{j+1/2},k,\overline{l+1/2}}$. Then the proposed explicit GA scheme can be written as

$$\mathbf{b}^{i+1} = \mathbf{b}^i - \mathbf{C}\mathbf{e}^{i+1/2} \tag{22}$$

$$\begin{aligned}\mathbf{e}^{i+3/2} &= \mathbf{e}^{i+1/2} - \Delta t \mathbf{M}_\varepsilon^{-1}\mathbf{j}_e^{i+1} \\ &+ (\Delta t)^2 \mathbf{M}_\varepsilon^{-1}\tilde{\mathbf{C}}\mathbf{M}_\mu^{-1}\mathbf{b}^{i+1}.\end{aligned} \tag{23}$$

with material matrices $\mathbf{M}_\sigma$, $\mathbf{M}_\varepsilon$ and $\mathbf{M}_\mu$ and curl operators $\mathbf{C}$, $\tilde{\mathbf{C}}$. They contain entries $-1, 0, 1$ for constructing the sums of (17) and (18) in matrix form.
On the other hand, we recall the equations for FIT with Leap-Frog time integration; they read

$$\widehat{\widehat{\mathbf{b}}}^{i+1} = \widehat{\widehat{\mathbf{b}}}^{i} - \Delta t \mathbf{C}\widehat{\mathbf{e}}^{i+1/2} \tag{24}$$

$$\begin{aligned}\widehat{\mathbf{e}}^{i+3/2} &= \widehat{\mathbf{e}}^{i+1/2} - \Delta t \mathbf{M}_\varepsilon^{-1}\widehat{\widehat{\mathbf{j}}}_e^{i+1} \\ &+ \Delta t \mathbf{M}_\varepsilon^{-1}\tilde{\mathbf{C}}\mathbf{M}_\mu^{-1}\widehat{\widehat{\mathbf{b}}}^{i+1}.\end{aligned} \tag{25}$$

As previously shown, in the context of our GA scheme we use

$$\frac{e_{\overline{i+1/2},\overline{j+1/2},k,l}}{\Delta t} \text{ instead of } e_{i+1/2,\overline{j+1/2},k,l}, \tag{26}$$

$$b_{i,j,\overline{k+1/2},\overline{l+1/2}} \text{ instead of } b_{i,j,\overline{k+1/2},\overline{l+1/2}},$$

$$\frac{j_{\bar{i},j+1/2,\bar{k},\bar{l}}}{\Delta t} \text{ instead of } j_{i,j+1/2,\bar{k},\bar{l}}, \tag{27}$$

which can be written symbolically as

$$\mathbf{e}^i \mathrel{\hat{=}} \Delta t \widehat{\mathbf{e}}^i, \qquad \mathbf{b}^i \mathrel{\hat{=}} \widehat{\widehat{\mathbf{b}}}^i, \qquad \mathbf{j}_e^i \mathrel{\hat{=}} \Delta t \widehat{\widehat{\mathbf{j}}}_e^i,$$

where $\widehat{\widehat{\mathbf{b}}}$, $\widehat{\mathbf{e}}$ and $\widehat{\widehat{\mathbf{j}}}$ are vectors created from FIT degrees of freedom. However, it is sufficient to define the initial value for the electric field adequately as

$$\mathbf{e}^{1/2} = \Delta t \widehat{\mathbf{e}}^{1/2}. \tag{28}$$

Then, from (24) and (22) we see that

$$\mathbf{b}^1 = \mathbf{b}^0 - \mathbf{C}\mathbf{e}^{1/2} = \widehat{\widehat{\mathbf{b}}}^0 - \Delta t \widehat{\mathbf{e}}^{1/2} = \widehat{\widehat{\mathbf{b}}}^1$$

and by assuming (27):

$$\begin{aligned}\mathbf{e}^{3/2} &= \mathbf{e}^{1/2} - \Delta t \mathbf{M}_\varepsilon^{-1}\mathbf{j}_e^1 + (\Delta t)^2 \mathbf{M}_\varepsilon^{-1}\tilde{\mathbf{C}}\mathbf{M}_\mu^{-1}\mathbf{b}^1 \\ &= \Delta t \widehat{\mathbf{e}}^{1/2} - \Delta t^2 \mathbf{M}_\varepsilon^{-1}\widehat{\widehat{\mathbf{j}}}_e^1 \\ &+ (\Delta t)^2 \mathbf{M}_\varepsilon^{-1}\tilde{\mathbf{C}}\mathbf{M}_\mu^{-1}\widehat{\widehat{\mathbf{b}}}^1 = \Delta t \widehat{\mathbf{e}}^{3/2}.\end{aligned}$$

Therefore, assuming (28), which refers to the initial time step, the (26), which refers to all time steps, is true.

VI. Conclusion

Although FIT is obtained from (3) and (4) while the finite-difference-like GA are derived from (1) and (2), the corresponding numerical schemes are the same (with minor differences described above). Furthermore our GA scheme is equivalent to Tonti's approach for a specific treatment of Ohm's Law.

It might be surprising that the space-time integral formulation does not produce a different scheme than that using a classical space discretization with Leap-Frog, i.e., an explicit time integration scheme, but both methods are based on midpoint quadrature. To conclude, the schemes are closely related: if we have a FIT solver, the only modification we need to carry out is to change FIT's current definition

$$j_{i,j+1/2,\bar{k},\bar{l}} = \int dydz J_e^x$$

to its value averaged over one time step

$$j_{i,j+1/2,\bar{k},\bar{l}} = \frac{1}{\Delta t} j_{\bar{i},j+1/2,\bar{k},\bar{l}} = \left(\frac{1}{\Delta t}\int dt\right)\int dydz J_e^x$$

to obtain a solver for the finite-difference approach to Maxwell's equation in the space-time GA setting.

Acknowledgment

The work of the first and third author is supported by the 'Excellence Initiative' of the German Federal and State Governments and the Graduate School of Computational Engineering at Technische Universitaet Darmstadt. The second author is supported by the DFG through SFB 634.

References

[1] Hestenes, *Space-Time Algebra*. New York: Gordon and Breach, 1966.

[2] (2007) Geometric calculus website. [Online]. Available: http://geocalc.clas.asu.edu/

[3] C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, 2nd ed. Cambridge: Cambridge University Press, 2003.

[4] L. Dorst, D. Fontijne, and S. Mann, *Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry*, ser. The Morgan Kaufmann Series in Computer Graphics. San Francisco: Morgan Kaufmann Publishers Inc., 2007.

[5] D. Hildenbrand, *Foundations of Geometric Algebra Computing*, ser. Geometry and Computing Series. Springer-Verlag, 2012.

[6] K. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Transactions on Antennas and Propagation*, vol. 14, pp. 302–307, May 1966.

[7] E. Tonti, "Finite formulation of the electromagnetic field," *PIER*, vol. 32, pp. 1–44, 2001.

[8] T. Weiland, "A discretization model for the solution of Maxwell's equations for six-component fields," *Archiv fuer Elektronik und Uebertragungstechnik*, vol. 31, pp. 116–120, Mar. 1977.

[9] ——, "Time domain electromagnetic field computation with finite difference methods," *International Journal of Numerical Modelling*, vol. 9, pp. 295–319, Jul. 1996.